Today

Facility Location.
Today

Facility Location.
Lagrangian Dual.
Today

Facility Location.

Lagrangian Dual. Already.
Today

Facility Location.
Lagrangian Dual. Already.
Convex Separator.
Today

Facility Location.
Lagrangian Dual. Already.
Convex Separator.
Farkas Lemma.
Facility location

Set of facilities: $F$, opening cost $f_i$ for facility $i$
Facility location

Set of facilities: $F$, opening cost $f_i$ for facility $i$
Set of clients: $D$. 
Facility location

Set of facilities: $F$, opening cost $f_i$ for facility $i$
Set of clients: $D$.

$d_{ij}$ - distance between $i$ and $j$. 

(Notation abuse: clients/facility confusion.)

Triangle inequality: $d_{ij} \leq d_{ik} + d_{kj}$. 
Facility location

Set of facilities: $F$, opening cost $f_i$ for facility $i$
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Facility location

Set of facilities: $F$, opening cost $f_i$ for facility $i$
Set of clients: $D$.

$d_{ij}$ - distance between $i$ and $j$.
(notation abuse: clients/facility confusion.)

Triangle inequality: $d_{ij} \leq d_{ik} + d_{kj}$. 

Facility Location

Linear program relaxation:

- Decision Variables:
  - $y_i$: facility $i$ open?
  - $x_{ij}$: client $j$ assigned to facility $i$.

Objective:
- Minimize: $\sum_{i \in F} f_i y_i + \sum_{i \in F, j \in D} d_{ij} x_{ij}$, $\forall j \in D$

Constraint 1:
- $\sum_{i \in F} x_{ij} \geq 1$, $\forall i \in F, j \in D$

Constraint 2:
- $x_{ij} \leq y_i$, $\forall i \in F, j \in D$

Constraints:
- $x_{ij}, y_i \geq 0$
Facility Location

Linear program relaxation:

“Decision Variables”.

\[
\min \sum_{i \in F} f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \quad \forall j \in D
\]

\[
\sum_{i \in F} x_{ij} \geq 1 \quad \forall i \in F, j \in D
\]

\[
x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0
\]

Facility opening cost.
Client connection cost.
Must connect each client.
Only connect to open facility.
Facility Location

Linear program relaxation:

“Decision Variables”.

\( y_i \) - facility i open?
Facility Location

Linear program relaxation:

“Decision Variables”.

\[ y_i \] - facility i open?

\[ x_{ij} \] - client j assigned to facility i.
Facility Location

Linear program relaxation:

“Decision Variables”.

\[ y_i \] - facility \( i \) open?
\[ x_{ij} \] - client \( j \) assigned to facility \( i \).
Facility Location

Linear program relaxation:

“Decision Variables”.

$y_i$ - facility $i$ open?

$x_{ij}$ - client $j$ assigned to facility $i$.

$$\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}$$

$$\forall j \in D \sum_{i \in F} x_{ij} \geq 1$$

$$\forall i \in F, j \in D \quad x_{ij} \leq y_i,$$

$$x_{ij}, y_i \geq 0$$
Facility Location

Linear program relaxation:

“Decision Variables”.

\( y_i \) - facility \( i \) open?

\( x_{ij} \) - client \( j \) assigned to facility \( i \).

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\( \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \)

\( \forall i \in F, j \in D \ x_{ij} \leq y_i \),

\( x_{ij}, y_i \geq 0 \)

Facility opening cost.
Facility Location

Linear program relaxation:

“Decision Variables”.
- \( y_i \) - facility \( i \) open?
- \( x_{ij} \) - client \( j \) assigned to facility \( i \).

\[
\begin{align*}
\text{min} & \quad \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{s.t.} & \quad \forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \\
\text{and} & \quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0
\end{align*}
\]

Facility opening cost.
Client Connection cost.
Facility Location

Linear program relaxation:

“Decision Variables”.

- $y_i$ - facility $i$ open?
- $x_{ij}$ - client $j$ assigned to facility $i$.

$$\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}$$

$$\forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1$$

$$\forall i \in F, j \in D \quad x_{ij} \leq y_i,$$

$$x_{ij}, y_i \geq 0$$

Facility opening cost.
Client Connection cost.
**Must connect each client.**
Facility Location

Linear program relaxation:

“Decision Variables”.

\( y_i \) - facility i open?

\( x_{ij} \) - client \( j \) assigned to facility \( i \).

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\( \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \)

\( \forall i \in F, j \in D \ x_{ij} \leq y_i, \)

\( x_{ij}, y_i \geq 0 \)

Facility opening cost.
Client Connection cost.
Must connect each client.
Only connect to open facility.
Integer Solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[
\forall j \in D \sum_{i \in F} x_{ij} \geq 1
\]

\[
\forall i \in F, j \in D \quad x_{ij} \leq y_i,
\]

\[
x_{ij}, y_i \geq 0
\]

\[
x_{ij} = \frac{1}{2} \text{ edges.}
\]

\[
y_i = \frac{1}{2} \text{ edges.}
\]
Integer Solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[
\forall j \in D \sum_{i \in F} x_{ij} \geq 1
\]

\[
\forall i \in F, j \in D \ x_{ij} \leq y_i, \\
\]

\[
x_{ij}, y_i \geq 0
\]

\[
x_{ij} = \frac{1}{2} \text{ edges.} \\
y_i = \frac{1}{2} \text{ edges.}
\]

Facility Cost: \(\frac{3}{2}\)
Integer Solution?

\[
\begin{align*}
\text{min} & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{subject to} & \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
& \forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

\[x_{ij} = \frac{1}{2} \text{ edges.}\]
\[y_i = \frac{1}{2} \text{ edges.}\]

Facility Cost: \(\frac{3}{2}\)
Connection Cost: 3
Integer Solution?

\[
\begin{align*}
\text{min} & \quad \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
& \quad \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
& \quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
& \quad x_{ij}, y_i \geq 0
\end{align*}
\]

\(x_{ij} = \frac{1}{2}\) edges.
\(y_i = \frac{1}{2}\) edges.

Facility Cost: \(\frac{3}{2}\)  
Connection Cost: 3

Any one Facility:
Facility Cost: 1
Integer Solution?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D & \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D & x_{ij} \leq y_i, \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

\[x_{ij} = \frac{1}{2} \text{ edges.}\]
\[y_i = \frac{1}{2} \text{ edges.}\]

Facility Cost: \(\frac{3}{2}\) Connection Cost: 3

Any one Facility:
Facility Cost: 1 Client Cost: 3.7
Integer Solution?

\[
\begin{align*}
&\text{min} \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
&\quad \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
&\quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
&\quad x_{ij}, y_i \geq 0
\end{align*}
\]

\(x_{ij} = \frac{1}{2}\) edges.  \\
\(y_i = \frac{1}{2}\) edges.  \\
Facility Cost: \(\frac{3}{2}\)  Connection Cost: 3  \\
Any one Facility:  \\
\quad Facility Cost: 1  Client Cost: 3.7  \\
Make it worse?
Integer Solution?

\[
\begin{align*}
\text{min} & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D & \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D & x_{ij} \leq y_i, \\
x_{ij}, y_i & \geq 0
\end{align*}
\]

\(x_{ij} = \frac{1}{2}\) edges.
\(y_i = \frac{1}{2}\) edges.

Facility Cost: \(\frac{3}{2}\) Connection Cost: 3
Any one Facility:
Facility Cost: 1 Client Cost: 3.7
Make it worse? Sure.
Integer Solution?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{ s.t. } & \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
& \forall i \in F, j \in D \ x_{ij} \leq y_i, \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

\(x_{ij} = \frac{1}{2}\) edges.
\(y_i = \frac{1}{2}\) edges.

Facility Cost: \(\frac{3}{2}\) Connection Cost: 3

Any one Facility:
Facility Cost: 1 Client Cost: 3.7

Make it worse? Sure. Not as pretty!
Round solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[\forall j \in D \sum_{i \in F} x_{ij} \geq 1\]

\[\forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0\]
Round solution?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{s.t.} & \quad \forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \\
\text{and} & \quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
& \quad x_{ij}, y_i \geq 0
\end{align*}
\]

Round independently?
Round independently?

$y_i$ and $x_{ij}$ separately?
Round solution?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D & \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D & x_{ij} \leq y_i, \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

Round independently?

\(y_i\) and \(x_{ij}\) separately? Assign to closed facility!
Round solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[\forall j \in D \sum_{i \in F} x_{ij} \geq 1\]

\[\forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0\]

Round independently?

\(y_i\) and \(x_{ij}\) separately? Assign to closed facility!

Round \(x_{ij}\) and open facilities?
Round solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[\forall j \in D \sum_{i \in F} x_{ij} \geq 1\]

\[\forall i \in F, j \in D \quad x_{ij} \leq y_i,\quad x_{ij}, y_i \geq 0\]

Round independently?

\(y_i\) and \(x_{ij}\) separately? Assign to closed facility!

Round \(x_{ij}\) and open facilities?

Different clients force different facilities open.
Round solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[
\forall j \in D \sum_{i \in F} x_{ij} \geq 1
\]

\[
\forall i \in F, j \in D \ x_{ij} \leq y_i,
\]

\[
x_{ij}, y_i \geq 0
\]

Round independently?

\(y_i\) and \(x_{ij}\) separately? Assign to closed facility!

Round \(x_{ij}\) and open facilities?

Different clients force different facilities open.

Any ideas?
Round solution?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}
\]

\[\forall j \in D \sum_{i \in F} x_{ij} \geq 1\]

\[\forall i \in F, j \in D \quad x_{ij} \leq y_i,\]

\[x_{ij}, y_i \geq 0\]

Round independently?

\[y_i\] and \[x_{ij}\] separately? Assign to closed facility!

Round \[x_{ij}\] and open facilities?

Different clients force different facilities open.

Any ideas?

Use Dual!
The dual.

\[ \min cx, Ax \geq b \]
The dual.

$$\min cx, Ax \geq b \leftrightarrow$$
The dual.

\[
\min cx, \ Ax \geq b \leftrightarrow \max bx, \ y^T A \leq c.
\]
The dual.

\[ \min cx, Ax \geq b \iff \max bx, y^T A \leq c. \]

\[ \min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \]

\[ \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \]

\[ \forall i \in F, j \in D \quad x_{ij} \leq y_i, \]
The dual.

\[ \min cx, Ax \geq b \leftrightarrow \max bx, y^T A \leq c. \]

\[ \begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D & \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D & x_{ij} \leq y_i,
\end{align*} \]
The dual.

\[ \min cx, Ax \geq b \iff \max bx, y^T A \leq c. \]

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D & \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D & x_{ij} \leq y_i,
\end{align*}
\]
Interpretation of Dual?

\[
\begin{align*}
\min & \quad \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{subject to} & \quad \forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \\
& \quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
& \quad x_{ij}, y_i \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad \sum_{j} \alpha_j \\
\text{subject to} & \quad \forall i \in F \quad \sum_{j \in D} \beta_{ij} \leq f_i \\
& \quad \forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} x_{ij} \\
& \quad \alpha_j, \beta_{ij} \leq 0
\end{align*}
\]
Interpretation of Dual?

\[
\begin{align*}
\text{min} & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\quad & \forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \\
\quad & \forall i \in F, j \in D \quad x_{ij} \leq y_i, x_{ij}, y_i \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \sum_j \alpha_j \\
\quad & \forall i \in F \quad \sum_{j \in D} \beta_{ij} \leq f_i \\
\quad & \forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} \\
\quad & \alpha_j, \beta_{ij} \leq 0
\end{align*}
\]

\(\alpha_j\) charge to client.
Interpretation of Dual?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{s.t.} & \quad \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
& \quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
& \quad x_{ij}, y_i \geq 0
\end{align*}
\]

\[\alpha_j \text{ charge to client.} \]

maximize price paid by client to connect!

\[
\begin{align*}
\max & \sum_j \alpha_j \\
\text{s.t.} & \quad \forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i \\
& \quad \forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} x_{ij} \\
& \quad \alpha_j, \beta_{ij} \leq 0
\end{align*}
\]
Interpretation of Dual?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{subject to} & \sum_{j \in D} x_{ij} \geq 1, & \forall i \in F \\
& \sum_{i \in F} x_{ij} \leq y_i, & \forall i \in F, j \in D \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

\[\alpha_j \text{ charge to client.}
max \sum_{j} \alpha_j
\]

\[\forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i
\]

\[\forall i \in F, j \in D \alpha_j - \beta_{ij} \leq d_{ij} \]

\[x_{ij} \leq \alpha_j, \beta_{ij} \leq 0
\]

\[\text{maximize price paid by client to connect!}
\]

Objective: \(\sum_j \alpha_j\) total payment.
Interpretation of Dual?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
\alpha_j, y_i \geq 0
\]

\[
\max \sum_{j} \alpha_j \\
\forall i \in F \quad \sum_{j \in D} \beta_{ij} \leq f_i \\
\forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} \quad x_{ij} \\
\alpha_j, \beta_{ij} \leq 0
\]

\(\alpha_j\) charge to client.

maximize price paid by client to connect!

Objective: \(\sum_j \alpha_j\) total payment.

Client \(j\) travels or pays to open facility \(i\).
**Interpretation of Dual?**

\[
\begin{align*}
\min & \sum_{i \in F} y_if_i + \sum_{i \in F, j \in D} x_{ij}d_{ij} \\
\text{subject to} & \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
& \forall i \in F, j \in D \\ & x_{ij} \leq y_i, \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \sum_j \alpha_j \\
\text{subject to} & \forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i \\
& \forall i \in F, j \in D \\ & \alpha_j - \beta_{ij} \leq d_{ij} \\ & \alpha_j, \beta_{ij} \leq 0
\end{align*}
\]

\(\alpha_j\) charge to client.
maximize price paid by client to connect!

Objective: \(\sum_j \alpha_j\) total payment.
Client \(j\) travels or pays to open facility \(i\).
Costs client \(d_{ij}\) to get to there.

---

\(x_{ij}\), \(y_i\) ≥ 0

---
Interpretation of Dual?

\[ \min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \]

\[ \forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \]

\[ \forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0 \]

\[ \alpha_j \] charge to client.

maximize price paid by client to connect!

Objective: \( \sum_j \alpha_j \) total payment.

Client \( j \) travels or pays to open facility \( i \).

Costs client \( d_{ij} \) to get to there.

Savings is \( \alpha_j - d_{ij} \).
Interpretation of Dual?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{subject to} & \quad \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
\text{subject to} & \quad \forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0
\end{align*}
\]

\(\alpha_j\) charge to client.
maximize price paid by client to connect!

Objective: \(\sum_j \alpha_j\) total payment.

Client \(j\) travels or pays to open facility \(i\).
Costs client \(d_{ij}\) to get to there.
Savings is \(\alpha_j - d_{ij}\).
Willing to pay \(\beta_{ij} = \alpha_j - d_{ij}\).

\[
\begin{align*}
\max & \sum_j \alpha_j \\
\text{subject to} & \quad \forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i \\
\text{subject to} & \quad \forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} \quad x_{ij} \\
\text{subject to} & \quad \alpha_j, \beta_{ij} \leq 0
\end{align*}
\]
**Interpretation of Dual?**

\[
\begin{align*}
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D \quad x_{ij} \leq y_i, \\
x_{ij}, y_i \geq 0
\end{align*}
\]

\[
\begin{align*}
\max \sum_j \alpha_j \\
\forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i \\
\forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} x_{ij} \\
\alpha_j, \beta_{ij} \leq 0
\end{align*}
\]

\(\alpha_j\) charge to client.
maximize price paid by client to connect!

**Objective:** \(\sum_j \alpha_j\) total payment.

Client \(j\) travels or pays to open facility \(i\).
Costs client \(d_{ij}\) to get to there.
Savings is \(\alpha_j - d_{ij}\).
Willing to pay \(\beta_{ij} = \alpha_j - d_{ij}\).

Total payment to facility \(i\) at most \(f_i\) before opening.
Interpretation of Dual?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{subject to} & \sum_{i \in F} x_{ij} \geq 1 \quad \forall j \in D \\
& \sum_{i \in F} x_{ij} \leq y_i, \quad \forall i \in F, j \in D \\
& x_{ij}, y_i \geq 0
\end{align*}
\]

\(\alpha_j\) charge to client.

maximize price paid by client to connect!

Objective: \(\sum_j \alpha_j\) total payment.

Client \(j\) travels or pays to open facility \(i\).

Costs client \(d_{ij}\) to get to there.

Savings is \(\alpha_j - d_{ij}\).

Willing to pay \(\beta_{ij} = \alpha_j - d_{ij}\).

Total payment to facility \(i\) at most \(f_i\) before opening.

Complementary slackness:

\[
\begin{align*}
\left( x_{ij} \right) \geq 0 & \text{ if and only if } \alpha_{j} \geq d_{ij} \\
\beta_{ij} & \leq 0
\end{align*}
\]
Interpretation of Dual?

\[
\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
\forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0
\]

\[\alpha_j\] charge to client.

maximize price paid by client to connect!

Objective: \(\sum_j \alpha_j\) total payment.

Client \(j\) travels or pays to open facility \(i\).

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Total payment to facility \(i\) at most \(f_i\) before opening.

Complementary slackness: \(x_{ij} \geq 0\) if and only if \(\alpha_j \geq d_{ij}\).
Interpretation of Dual?

\[
\begin{align*}
\min & \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij} \\
\text{subject to} & \forall j \in D \sum_{i \in F} x_{ij} \geq 1 \\
& \forall i \in F, j \in D \quad x_{ij} \leq y_i, \quad x_{ij}, y_i \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \sum_j \alpha_j \\
\text{subject to} & \forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i \\
& \forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} \quad x_{ij} \\
& \alpha_j, \beta_{ij} \leq 0
\end{align*}
\]

\(\alpha_j\) charge to client.
    maximize price paid by client to connect!

Objective: \(\sum_j \alpha_j\) total payment.
Client \(j\) travels or pays to open facility \(i\).
    Costs client \(d_{ij}\) to get to there.
    Savings is \(\alpha_j - d_{ij}\).
    Willing to pay \(\beta_{ij} = \alpha_j - d_{ij}\).

Total payment to facility \(i\) at most \(f_i\) before opening.
Complementary slackness: \(x_{ij} \geq 0\) if and only if \(\alpha_j \geq d_{ij}\).
    only assign client to “paid to” facilities.
Use Dual.

1. Find solution to primal, \((x, y)\). and dual, \((\alpha, \beta)\).
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2. For smallest (remaining) \(\alpha_j\),
Use Dual.

1. Find solution to primal, \((x, y)\). and dual, \((\alpha, \beta)\).

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   (a) Let \(N_j = \{i : x_{ij} > 0\}\).
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1. Find solution to primal, \((x, y)\). and dual, \((\alpha, \beta)\).

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1. Find solution to primal, \((x, y)\). and dual, \((\alpha, \beta)\).

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Integral facility cost at most LP facility cost.

Claim: Total facility cost is at most \( \sum_i f_i y_i \).
Integral facility cost at most LP facility cost.

**Claim:** Total facility cost is at most \( \sum_i f_i y_i \).

2. For smallest (remaining) \( \alpha_j \),
Integral facility cost at most LP facility cost.

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      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

**Proof:** Step 2 picks client $j$. 
Integral facility cost at most LP facility cost.

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2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
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      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

**Proof:** Step 2 picks client $j$.
$f_{\text{min}}$ - min cost facility in $N_j$
Integral facility cost at most LP facility cost.

**Claim:** Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
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   $f_{\text{min}}$ - min cost facility in $N_j$
   $f_{\text{min}}$
Integral facility cost at most LP facility cost.

Claim: Total facility cost is at most $\sum_i f_i y_i$.

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Proof: Step 2 picks client $j$.

$f_{\text{min}}$ - min cost facility in $N_j$

$f_{\text{min}} \leq f_{\text{min}} \cdot \sum_{i \in N_j} x_{ij}$
Integral facility cost at most LP facility cost.

**Claim:** Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

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$f_{\text{min}}$ - min cost facility in $N_j$

$f_{\text{min}} \leq f_{\text{min}} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\text{min}} \sum_{i \in N_j} y_i$
Integral facility cost at most LP facility cost.

Claim: Total facility cost is at most \( \sum_i f_i y_i \).

2. For smallest (remaining) \( \alpha_j \),
   (a) Let \( N_j = \{i : x_{ij} > 0\} \).
   (b) Open cheapest facility \( i \) in \( N_j \).
      Every client \( j' \) with \( N_j' \cap N_j \neq \emptyset \) assigned to \( i \).

Proof: Step 2 picks client \( j \).

\[
f_{\text{min}} - \text{min cost facility in } N_j \\ f_{\text{min}} \leq f_{\text{min}} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\text{min}} \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i.
\]
Integral facility cost at most LP facility cost.

**Claim:** Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_j' \cap N_j \neq \emptyset$ assigned to $i$.

**Proof:** Step 2 picks client $j$.

$\min_i f_i$ - min cost facility in $N_j$

$f_{\min} \leq \min_i f_i \cdot \sum_{i \in N_j} x_{ij} \leq \min_i f_i \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i$.

For $k$ used in Step 2.
Integral facility cost at most LP facility cost.

**Claim:** Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

**Proof:** Step 2 picks client $j$.
$f_{\min} - \min$ cost facility in $N_j$

$$f_{\min} \leq f_{\min} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\min} \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i.$$  

For $k$ used in Step 2.
$N_j \cap N_k = \emptyset$ for $j$ and $k$ in step 2.
**Integral facility cost at most LP facility cost.**

**Claim:** Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

**Proof:** Step 2 picks client $j$.
$f_{\min} - \min$ cost facility in $N_j$

$$f_{\min} \leq f_{\min} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\min} \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i.$$  

For $k$ used in Step 2.
$N_j \cap N_k = \emptyset$ for $j$ and $k$ in step 2.
$\rightarrow$ Any facility in $\leq 1$ sum from step 2.
Integral facility cost at most LP facility cost.

**Claim:** Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

**Proof:** Step 2 picks client $j$.

$f_{\text{min}}$ - min cost facility in $N_j$

$f_{\text{min}} \leq f_{\text{min}} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\text{min}} \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i$.

For $k$ used in Step 2.  
$N_j \cap N_k = \emptyset$ for $j$ and $k$ in step 2.  
$\rightarrow$ Any facility in $\leq 1$ sum from step 2.  
$\rightarrow$ total step 2 facility cost is $\sum_i y_i f_i$.  

Claim: Total facility cost is at most $\sum_i f_i y_i$.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
       Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Proof: Step 2 picks client $j$.

$f_{\min} - \text{min cost facility in } N_j$

$f_{\min} \leq f_{\min} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\min} \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i$.

For $k$ used in Step 2.

$N_j \cap N_k = \emptyset$ for $j$ and $k$ in step 2.

$\rightarrow$ Any facility in $\leq 1$ sum from step 2.

$\rightarrow$ total step 2 facility cost is $\sum_i y_i f_i$. 
Connection Cost.

2. For smallest (remaining) $\alpha_j$, 

\[ N_j = \{ i : x_{ij} > 0 \} \]

(b) Open cheapest facility $i$ in $N_j$. Every client $j'$ with $N_{j'} \cap N_j \neq 0$ assigned to $i$. Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$:

\[ \leq \alpha_j \]

Connection Cost of $j'$:

\[ \leq \alpha_j' + \alpha_j + \alpha_j' \leq 3 \alpha_j' \]

since $\alpha_j \leq \alpha_j'$

Total connection cost:

\[ \leq 3 \sum j' \alpha_j' \leq 3 \text{ times Dual OPT.} \]

Previous Slide: Facility cost:

\[ \leq \text{primal "facility" cost} \leq \text{Primal OPT.} \]

Total Cost: 4 OPT.
Connection Cost.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
2. For smallest (remaining) \( \alpha_j \),
   (a) Let \( N_j = \{ i : x_{ij} > 0 \} \).
   (b) Open cheapest facility \( i \) in \( N_j \).

Connections Cost:

Connection Cost of \( j \):
\[ \leq \alpha_j \]

Connection Cost of \( j' \):
\[ \leq \alpha_j' + \alpha_j + \alpha_j' \leq 3 \alpha_j' \]
since \( \alpha_j \leq \alpha_j' \)

Total connection cost:
at most \[3 \sum j' \alpha_j' \leq 3 \times \text{Dual OPT.} \]
Connection Cost.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$. 

Client $j$ is directly connected. Clients $j'$ are indirectly connected. 
Connection Cost of $j$: $\leq \alpha_j$. 
Connection Cost of $j'$: $\leq \alpha_{j'} + \alpha_j \leq 3 \alpha_{j'}$. 

Total connection cost: at most $3 \sum_{j'} \alpha_{j'} \leq 3 \times \text{Dual OPT}$. 

Previous Slide: Facility cost: $\leq \text{primal "facility" cost} \leq \text{Primal OPT}$. 
Total Cost: $4 \times \text{OPT}$. 
2. For smallest (remaining) $\alpha_j$,  
   (a) Let $N_j = \{i : x_{ij} > 0\}$.  
   (b) Open cheapest facility $i$ in $N_j$.  
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.  

Connection Cost of $j$: $\leq \alpha_j$.  
Connection Cost of $j'$: $\leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3\alpha_{j'}$.  

Total connection cost: at most $3 \sum_j \alpha_{j'} \leq 3 \text{ times Dual OPT.}$

Previous Slide: Facility cost: $\leq$ primal "facility" cost $\leq$ Primal OPT.  
Total Cost: $4 \text{ OPT.}$
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_j \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.
Connection Cost.

2. For smallest (remaining) $\alpha_j$, 
   (a) Let $N_j = \{i : x_{ij} > 0\}$. 
   (b) Open cheapest facility $i$ in $N_j$. 
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$. 

Client $j$ is directly connected. Clients $j'$ are indirectly connected.
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: 

\[
\alpha_j \leq \alpha_{j'} + \alpha_j + \alpha_j' \leq 3 \alpha_j',
\] 

since $\alpha_j \leq \alpha_{j'}$.
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

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Connection Cost of $j$:
2. For smallest (remaining) $\alpha_j$, 
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       Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: \( \leq \alpha_j \).
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.

   Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

   Client $j$ is directly connected. Clients $j'$ are indirectly connected.

   Connection Cost of $j$: $\leq \alpha_j$.
   Connection Cost of $j'$:
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{ i : x_{ij} > 0 \}$.
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       Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: $\leq \alpha_j$.
Connection Cost of $j'$:
Connection Cost.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: $\leq \alpha_j$.
Connection Cost of $j'$: $\leq \alpha_{j'} + \alpha_j + \alpha_j$
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
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Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: \[ \leq \alpha_j. \]
Connection Cost of $j'$: \[ \leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3\alpha_{j'}. \]
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Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: $\leq \alpha_j$.
Connection Cost of $j'$:
$\leq \alpha_j + \alpha_j + \alpha_j \leq 3\alpha_{j'}$.
since $\alpha_j \leq \alpha_{j'}$
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.
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Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: $\leq \alpha_j$.
Connection Cost of $j'$:
$\leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3\alpha_{j'}$.
since $\alpha_j \leq \alpha_{j'}$

Total connection cost:
at most $3 \sum_{j'} \alpha_j \leq 3$ times Dual OPT.
Connection Cost.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility $i$ in $N_j$.

   Every client $j'$ with $N_{ji'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: $\leq \alpha_j$.

Connection Cost of $j'$:

$\leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3\alpha_{j'}$.

since $\alpha_j \leq \alpha_{j'}$

Total connection cost:

at most $3 \sum_{j'} \alpha_j \leq 3 \text{ times Dual OPT.}$

Previous Slide: Facility cost:
2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{ i : x_{ij} > 0 \}$.
   (b) Open cheapest facility $i$ in $N_j$.
   Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

Connection Cost of $j$: \[ \leq \alpha_j. \]
Connection Cost of $j'$:
\[ \leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3\alpha_{j'}. \]
   since $\alpha_j \leq \alpha_{j'}$

Total connection cost:
at most $3 \sum_{j'} \alpha_j \leq 3 \text{ times Dual OPT}$.

Previous Slide: Facility cost:
\[ \leq \text{primal “facility” cost} \leq \text{Primal OPT}. \]
Connection Cost.

2. For smallest (remaining) $\alpha_j$,
   (a) Let $N_j = \{ i : x_{ij} > 0 \}$.
   (b) Open cheapest facility $i$ in $N_j$.
      Every client $j'$ with $N_{j'} \cap N_j \neq \emptyset$ assigned to $i$.

Client $j$ is directly connected. Clients $j'$ are indirectly connected.

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Connection Cost of $j'$:
$\leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3 \alpha_{j'}$.
since $\alpha_j \leq \alpha_{j'}$

Total connection cost:
at most $3 \sum j' \alpha_j \leq 3 \text{ times Dual OPT}$.

Previous Slide: Facility cost:
$\leq \text{primal “facility” cost} \leq \text{Primal OPT}$.

Total Cost: 4 OPT.
Twist on randomized rounding.

Client $j$:
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$,  

Expected opening cost:

$$\sum_{i \in N} x_{ij} f_i \leq \sum_{i \in N} y_i f_i.$$  

Connection cost of primal for $j$:

$$D_j = \sum_i x_{ij} d_{ij}.$$  

Expected connection cost $j'$:

$$\alpha_j + \alpha_j' + D_j.$$  

In step 2: pick in increasing order of $\alpha_j + D_j$.  

Expected cost is $(2\alpha_j' + D_j')$. Connection cost: $2\sum_j \alpha_j + \sum_j D_j$.  

$2\text{OPT}(\mathcal{D})$ plus connection cost or primal.  

Total expected cost: Facility cost is at most facility cost of primal. Connection cost at most $2\text{OPT} + \text{connection cost of primal}$.  

$\rightarrow$ at most $3\text{OPT}$.  

Twist on randomized rounding.

Client $j$: $\sum x_{ij} = 1$, $x_{ij} \geq 0$. 
Twist on randomized rounding.

Client \( j \): \( \sum_i x_{ij} = 1, \ x_{ij} \geq 0 \).
Probability distribution!
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.
Probability distribution! $\rightarrow$ Choose from distribution, $x_{ij}$, in step 2.
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.

Probability distribution! $\rightarrow$ Choose from distribution, $x_{ij}$, in step 2.

Expected opening cost:

$$\sum_{i \in N} j x_{i} f_{i} \leq \sum_{i \in N} j y_{i} f_{i}.$$
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.

Probability distribution! $\rightarrow$ Choose from distribution, $x_{ij}$, in step 2.

Expected opening cost:
$\sum_{i \in N_j} x_{ij} f_i$

$\sum_i x_{ij} d_{ij}$

Connection cost of primal for $j$.

Expected connection cost $j' \alpha_j + \alpha_j' + D_j$.

In step 2: pick in increasing order of $\alpha_j + D_j$.

$\rightarrow$ Expected cost is $(2 \alpha_j' + D_j')$.

Connection cost: $2 \sum_j \alpha_j + \sum_j D_j$.

$2 \text{OPT}(D) + \text{connection cost of primal}$. 

Total expected cost: Facility cost is at most facility cost of primal.

Connection cost at most $2 \text{OPT} + \text{connection cost of primal}$. 

$\rightarrow$ at most $3 \text{OPT}$.
Twist on randomized rounding.

Client \( j \): \( \sum_i x_{ij} = 1, \ x_{ij} \geq 0. \)

Probability distribution! → Choose from distribution, \( x_{ij} \), in step 2.

Expected opening cost: \( \sum_{i \in N_j} x_{ij} f_i \leq \sum_{i \in N_j} y_i f_i. \)
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.

Probability distribution! $\rightarrow$ Choose from distribution, $x_{ij}$, in step 2.

Expected opening cost:

$$\sum_{i \in N_j} x_{ij} f_i \leq \sum_{i \in N_j} y_i f_i.$$ 
and separate balls implies total $\leq \sum_i y_i f_i$. 

$D_j = \sum_i x_{ij} d_{ij}$

Connection cost of primal for $j$.

Expected connection cost $j' \alpha_j + \alpha_j' + D_j$.

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$2 \text{OPT}(D)$ plus connection cost or primal.

Total expected cost: 
Facility cost is at most facility cost of primal.
Connection cost at most $2 \text{OPT} +$ connection cost of primal.

$\rightarrow$ at most $3 \text{OPT}$. 
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.

Probability distribution! $\rightarrow$ Choose from distribution, $x_{ij}$, in step 2.

Expected opening cost:
\[
\sum_{i \in N_j} x_{ij} f_i \leq \sum_{i \in N_j} y_i f_i.
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$D_j = \sum_i x_{ij}d_{ij}$ Connection cost of primal for $j$. 

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  and separate balls implies total $\leq \sum_i y_i f_i$.

$D_j = \sum_i x_{ij} d_{ij}$  Connection cost of primal for $j$.

Expected connection cost $j'$
Twist on randomized rounding.

Client $j$: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.

Probability distribution! $\rightarrow$ Choose from distribution, $x_{ij}$, in step 2.

Expected opening cost:

$\sum_{i \in N_j} x_{ij} f_i \leq \sum_{i \in N_j} y_i f_i$.

and separate balls implies total $\leq \sum_i y_i f_i$.

$D_j = \sum_i x_{ij} d_{ij}$  

Connection cost of primal for $j$.

Expected connection cost $j'$ $\alpha_j + \alpha_{j'} + D_j$. 

$2 \text{OPT}(D)$ plus connection cost or primal.

Total expected cost: Facility cost is at most facility cost of primal.

Connection cost at most $2 \text{OPT} +$ connection cost of primal.

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Twist on randomized rounding.

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$2OPT(D)$ plus connection cost cost or primal.

Total expected cost:
Facility cost is at most facility cost of primal.
Twist on randomized rounding.

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Total expected cost:
Facility cost is at most facility cost of primal.
Connection cost at most $2OPT +$ connection cost of primal.
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Connection cost of primal for $j$.

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$2OPT(D)$ plus connection cost or primal.

Total expected cost:

- Facility cost is at most facility cost of primal.
- Connection cost at most $2OPT$ + connection cost of primal.

$\rightarrow$ at most $3OPT$. 
Primal dual algorithm.

1. Feasible integer solution.
Primal dual algorithm.

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2. Feasible dual solution.
Primal dual algorithm.

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3. Cost of integer solution $\leq \alpha$ times dual value.
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Just did it.
Primal dual algorithm.

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Just did it. Used linear program.
Primal dual algorithm.

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Just did it. Used linear program. Faster?
Primal dual algorithm.

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Typically.
Primal dual algorithm.

1. Feasible integer solution.
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3. Cost of integer solution $\leq \alpha$ times dual value.

Just did it. Used linear program. Faster?

Typically.
   Begin with feasible dual.
Primal dual algorithm.

1. Feasible integer solution.
2. Feasible dual solution.
3. Cost of integer solution ≤ \( \alpha \) times dual value.

Just did it. Used linear program. Faster?

Typically.
   Begin with feasible dual.
   Raise dual variables until tight constraint.
Primal dual algorithm.

1. Feasible integer solution.
2. Feasible dual solution.
3. Cost of integer solution $\leq \alpha$ times dual value.

Just did it. Used linear program. Faster?

Typically.

Begin with feasible dual.
Raise dual variables until tight constraint.
Set corresponding primal variable to an integer.
Primal dual algorithm.

1. Feasible integer solution.
2. Feasible dual solution.
3. Cost of integer solution $\leq \alpha$ times dual value.

Just did it. Used linear program. Faster?

Typically.
   - Begin with feasible dual.
   - Raise dual variables until tight constraint.
   - Set corresponding primal variable to an integer.

Recall Dual:

$$
\text{max } \sum_{j} \alpha_j \\
\forall i \in F \\
\sum_{j} \beta_{ij} \leq f_i \\
\forall i \in F, j \in D \\
\alpha_j - \beta_{ij} \leq d_{ij} \\
\alpha_j, \beta_{ij} \leq 0
$$
Primal dual algorithm.

1. Feasible integer solution.
2. Feasible dual solution.
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Just did it. Used linear program. Faster?

Typically.

Begin with feasible dual.

Raise dual variables until tight constraint.

Set corresponding primal variable to an integer.

Recall Dual:

$$\max \sum_j \alpha_j$$

$$\forall i \in F \sum_{j \in D} \beta_{ij} \leq f_i$$

$$\forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij}$$

$$\alpha_j, \beta_{ij} \leq 0$$
Facility location primal dual.

Phase 1:

1. Initially $\alpha_j, \beta_{ij} = 0$.
2. Raise $\alpha_j$ for every (unconnected) client. When $\alpha_j = d_{ij}$ for some $i$ raise $\beta_{ij}$ at same rate. Why?

Dual:

$\alpha_j - \beta_{ij} \leq d_{ij}$.

Intuition: Paying $\beta_{ij}$ to open $i$.

Stop when $\sum_i \beta_{ij} = f_i$. Why?

Dual:

$\sum_i \beta_{ij} \leq f_i$.

Intuition: Facility paid for.

Temporarily open $i$.

Connect all tight $ji$ clients $j$ to $i$.

3. Continue until all clients connected.

Phase 2:

Connect facilities that were paid by same client.

Permanently open an independent set of facilities.

For client $j$, connected facility $i$ is opened.

Good.

Connected facility not open $\rightarrow$ exists client $j'$ paid $i$ and connected to open facility.

Connect $j$ to $j'$'s open facility.
Facility location primal dual.

**Phase 1:** 1. Initially $\alpha_j, \beta_{ij} = 0$. 
Facility location primal dual.

**Phase 1:** 1. Initially $\alpha_j, \beta_{ij} = 0$.
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**Dual:** $\alpha_j - \beta_{ij} \leq d_{ij}$.
   *Intuition:* Paying $\beta_{ij}$ to open $i$.

Stop when $\sum_i \beta_{ij} = f_i$.
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Facility location primal dual.

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Temporarily open $i$.
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   - Intuition: facility paid for.

**Temporarily open $i$.**

**Connect all tight $ji$ clients $j$ to $i$.**

3. Continue until all clients connected.

**Phase 2:**
Connect facilities that were paid by same client.
Permanently open an independent set of facilities.
Facility location primal dual.

**Phase 1:**
1. Initially $\alpha_j, \beta_{ij} = 0$.
2. Raise $\alpha_j$ for every (unconnected) client.
   
   When $\alpha_j = d_{ij}$ for some $i$
   
   raise $\beta_{ij}$ at same rate  
   
   Why? Dual: $\alpha_j - \beta_{ij} \leq d_{ij}$.
   
   Intuition: Paying $\beta_{ij}$ to open $i$.
   
   Stop when $\sum_i \beta_{ij} = f_i$.
   
   Why? Dual: $\sum_i \beta_{ij} \leq f_i$
   
   Intuition: facility paid for.
   
   Temporarily open $i$.
   
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3. Stop when $\sum_i \beta_{ij} = f_i$.
   Why? Dual: $\sum_i \beta_{ij} \leq f_i$
   Intuition: facility paid for.
   Temporarily open $i$.
   Connect all tight $ji$ clients $j$ to $i$.

3. Continue until all clients connected.

**Phase 2:**
Connect facilities that were paid by same client.
Permanently open an independent set of facilities.
For client $j$, connected facility $i$ is opened.
Facility location primal dual.

**Phase 1:** 1. Initially $\alpha_j, \beta_{ij} = 0$.
2. Raise $\alpha_j$ for every (unconnected) client. When $\alpha_j = d_{ij}$ for some $i$
   
   - raise $\beta_{ij}$ at same rate  
   - Why? Dual: $\alpha_j - \beta_{ij} \leq d_{ij}$.
   
   - Intuition: Paying $\beta_{ij}$ to open $i$.

   Stop when $\sum_i \beta_{ij} = f_i$.
   
   - Why? Dual: $\sum_i \beta_{ij} \leq f_i$
   
   - Intuition: facility paid for.

   - **Temporarily open** $i$.
   - **Connect** all tight $ji$ clients $j$ to $i$.

3. Continue until all clients connected.

**Phase 2:**
Connect facilities that were paid by same client.
Permanently open an independent set of facilities.

For client $j$, connected facility $i$ is opened. Good.
Facility location primal dual.

**Phase 1:**
1. Initially $\alpha_j, \beta_{ij} = 0$.
2. Raise $\alpha_j$ for every (unconnected) client.
   When $\alpha_j = d_{ij}$ for some $i$
   raise $\beta_{ij}$ at same rate  
   Why? Dual: $\alpha_j - \beta_{ij} \leq d_{ij}$.
   Intuition: Paying $\beta_{ij}$ to open $i$.
3. Stop when $\sum_i \beta_{ij} = f_i$.
   Why? Dual: $\sum_i \beta_{ij} \leq f_i$
   Intuition: facility paid for.
   Temporarily open $i$.
   Connect all tight $ji$ clients $j$ to $i$.

3. Continue until all clients connected.

**Phase 2:**
Connect facilities that were paid by same client.
Permanently open an independent set of facilities.

For client $j$, connected facility $i$ is opened. Good.
Connected facility not open
Facility location primal dual.

**Phase 1:**
1. Initially $\alpha_j, \beta_{ij} = 0$.
2. Raise $\alpha_j$ for every (unconnected) client.
   When $\alpha_j = d_{ij}$ for some $i$
     raise $\beta_{ij}$ at same rate  Why? Dual: $\alpha_j - \beta_{ij} \leq d_{ij}$.
     Intuition: Paying $\beta_{ij}$ to open $i$.
   Stop when $\sum_i \beta_{ij} = f_i$.
     Why? Dual: $\sum_i \beta_{ij} \leq f_i$
     Intuition: facility paid for.
   Temporarily open $i$.
   Connect all tight $ji$ clients $j$ to $i$.

3. Continue until all clients connected.

**Phase 2:**
Connect facilities that were paid by same client.
Permanently open an independent set of facilities.

For client $j$, connected facility $i$ is opened. Good.
Connected facility not open
   $\rightarrow$ exists client $j'$ paid $i$ and connected to open facility.
**Phase 1:**
1. Initially $\alpha_j, \beta_{ij} = 0$.
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   Intuition: facility paid for.
   Temporarily open $i$.
   Connect all tight $ji$ clients $j$ to $i$.

3. Continue until all clients connected.

**Phase 2:**
Connect facilities that were paid by same client.
Permanently open an independent set of facilities.

For client $j$, connected facility $i$ is opened. Good.
Connected facility not open
   $\rightarrow$ exists client $j'$ paid $i$ and connected to open facility.
Connect $j$ to $j'$’s open facility.
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]

\[ \alpha_i - \beta_{ij} \leq d_{ij} \]

Grow \( \alpha_j \).

\[ \alpha_j = d_{ij} \]

Tight constraint:

\[ \alpha_j - \beta_{ij} \leq d_{ij} \]

Grow \( \beta_{ij} \) (and \( \alpha_j \)).

\[ \sum_j \beta_{ij} = f_i \] for all facilities.

Tight:

\[ \sum_j \beta_{ij} \leq f_i \]

LP Cost:

\[ \sum_j \alpha_j = 4.5 \]

Temporarily open all facilities.

Assign Clients to "paid to" open facility.

Connect facilities with client that pays both.

Open independent set.

Connect to "killer" client's facility.

Cost: 1 + 3.7 = 4.7.

A bit more than the LP cost.
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Constraints for dual.
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]

\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).

\[ \alpha_j = d_{ij}! \]
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
\( \alpha_j = d_{ij}! \)
Tight constraint:
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).

\[ \alpha_j = d_{ij}! \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij}. \)
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij}. \)

Grow \( \beta_{ij} \) (and \( \alpha_j \)).
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

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Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
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\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \text{ for all facilities.} \]
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
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\[ \sum_j \beta_{ij} = f_i \text{ for all facilities.} \]
Tight: \[ \sum_j \beta_{ij} \leq f_i \]
Constraints for dual.
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\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
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Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \text{ for all facilities.} \]
Tight: \( \sum_j \beta_{ij} \leq f_i \)

Cost: 1 + 3.7 = 4.7.
A bit more than the LP cost.
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]
Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).
Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \] for all facilities.
Tight: \( \sum_j \beta_{ij} \leq f_i \)
LP Cost: \( \sum_j \alpha_j \)
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]
Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij}. \)
Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \text{ for all facilities.} \]
Tight: \( \sum_j \beta_{ij} \leq f_i \)
LP Cost: \( \sum_j \alpha_j = 4.5 \)
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]
Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).
Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \] for all facilities.
Tight: \( \sum_j \beta_{ij} \leq f_i \)
LP Cost: \( \sum_j \alpha_j = 4.5 \)

Temporarily open all facilities.
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]
Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).
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\[ \sum_j \beta_{ij} = f_i \] for all facilities.
Tight: \( \sum_j \beta_{ij} \leq f_i \)
LP Cost: \( \sum_j \alpha_j = 4.5 \)

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Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
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Grow \( \alpha_j \).
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Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \text{ for all facilities.} \]

Tight: \[ \sum_j \beta_{ij} \leq f_i \]

LP Cost: \[ \sum_j \alpha_j = 4.5 \]

Temporarily open all facilities.
Assign Clients to “paid to” open facility.
β = .5

α = 1.5

Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \text{ for all facilities.} \]

Tight: \( \sum_j \beta_{ij} \leq f_i \)

LP Cost: \( \sum_j \alpha_j = 4.5 \)

Temporarily open all facilities.
Assign Clients to “paid to” open facility.
Connect facilities with client that pays both.
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).

\( \alpha_j = d_{ij} \)

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

Grow \( \beta_{ij} \) (and \( \alpha_j \)).

\[ \sum_j \beta_{ij} = f_i \] for all facilities.

Tight: \( \sum_j \beta_{ij} \leq f_i \)

LP Cost: \( \sum_j \alpha_j = 4.5 \)

Temporarily open all facilities.

Assign Clients to “paid to” open facility.

Connect facilities with client that pays both.

Open independent set.
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]
Grow \( \alpha_j \).
\[ \alpha_j = d_{ij}! \]
Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).
Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \text{ for all facilities}. \]
Tight: \( \sum_j \beta_{ij} \leq f_i \)
LP Cost: \( \sum_j \alpha_j = 4.5 \)

Temporarily open all facilities.
Assign Clients to “paid to” open facility.
Connect facilities with client that pays both.
Open independent set.
Connect to “killer” client’s facility.
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij} \]

Grow \( \alpha_j \).
\[ \alpha_j = d_{ij} \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \] for all facilities.

Tight: \( \sum_j \beta_{ij} \leq f_i \)

LP Cost: \( \sum_j \alpha_j = 4.5 \)

Temporarily open all facilities.

Assign Clients to “paid to” open facility.

Connect facilities with client that pays both.

Open independent set.

Connect to “killer” client’s facility.

Cost: 1 + 3.7
Constraints for dual.

\[ \sum_j \beta_{ij} \leq f_i \]
\[ \alpha_i - \beta_{ij} \leq d_{ij}. \]

Grow \( \alpha_j \).

\[ \alpha_j = d_{ij}! \]

Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).

Grow \( \beta_{ij} \) (and \( \alpha_j \)).

\[ \sum_j \beta_{ij} = f_i \] for all facilities.

Tight: \[ \sum_j \beta_{ij} \leq f_i \]

LP Cost: \( \sum_j \alpha_j = 4.5 \)

Temporarily open all facilities.

Assign Clients to “paid to” open facility.

Connect facilities with client that pays both.

Open independent set.

Connect to “killer” client’s facility.

Cost: \( 1 + 3.7 = 4.7. \)
Constraints for dual.
\[ \sum_j \beta_{ij} \leq f_i \]
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Grow \( \alpha_j \).
\( \alpha_j = d_{ij}! \)
Tight constraint: \( \alpha_j - \beta_{ij} \leq d_{ij} \).
Grow \( \beta_{ij} \) (and \( \alpha_j \)).
\[ \sum_j \beta_{ij} = f_i \] for all facilities.
Tight: \[ \sum_j \beta_{ij} \leq f_i \]
LP Cost: \[ \sum_j \alpha_j = 4.5 \]

Temporarily open all facilities.
Assign Clients to “paid to” open facility.
Connect facilities with client that pays both.
Open independent set.
Connect to “killer” client’s facility.
Cost: \[ 1 + 3.7 = 4.7. \]
A bit more than the LP cost.
Claim: Client only pays one facility.
Analysis

Claim: Client only pays one facility.

Independent set of facilities.
Analysis

**Claim:** Client only pays one facility.

Independent set of facilities.

**Claim:** $S_i$ - directly connected clients to open facility $i$. 
Analysis

Claim: Client only pays one facility.
Independent set of facilities.
Claim: $S_i$ - directly connected clients to open facility $i$.
\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]
Analysis

Claim: Client only pays one facility.

Independent set of facilities.

Claim: $S_i$ - directly connected clients to open facility $i$.

$$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j.$$

Proof:
Claim: Client only pays one facility.

Independent set of facilities.

Claim: $S_i$ - directly connected clients to open facility $i$.

$$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j.$$ 

Proof:

$$f_i = \sum_{j \in S_i} \beta_{ij}$$
Claim: Client only pays one facility. 

Independent set of facilities.

Claim: $S_i$ - directly connected clients to open facility $i$. 
\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]

Proof:
\[ f_i = \sum_{j \in S_i} \beta_{ij} = \sum_{j \in S_i} \alpha_j - d_{ij}. \]
Since directly connected: $\beta_{ij} = \alpha_j - d_{ij}$. 
}\]
Analysis.

**Claim:** Client $j$ is indirectly connected to $i$
**Claim:** Client $j$ is indirectly connected to $i$

$ightarrow d_{ij} \leq 3\alpha_j.$
Claim: Client $j$ is indirectly connected to $i$
$\rightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$
Analysis.

**Claim:** Client $j$ is indirectly connected to $i$
$\rightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$ conflicts with $i$. 
Claim: Client $j$ is indirectly connected to $i$
$\rightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$ conflicts with $i$.
exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$. 
**Claim:** Client $j$ is indirectly connected to $i$

\[ d_{ij} \leq 3 \alpha_j. \]

Directly connected to (temp open) $i'$ conflicts with $i$.

- exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.
- When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$. 
**Claim:** Client $j$ is indirectly connected to $i$  
$ightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$ conflicts with $i$.

exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_j'$.

$\alpha_j'$ stopped no later
Claim: Client $j$ is indirectly connected to $i$
$\rightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$
conflicts with $i$.
exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.
$\alpha_{j'}$ stopped no later (..maybe earlier..)
Claim: Client $j$ is indirectly connected to $i$

$\rightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$ conflicts with $i$.

exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.

$\alpha_{j'}$ stopped no later (..maybe earlier..)

$\alpha_j \leq \alpha_{j'}$. 

Total distance from $j$ to $i'\leq 3\alpha_j$. 

$d_{ij} + d_{ij'} + d_{i'j'} \leq 3\alpha_j$. 

Claim: Client $j$ is indirectly connected to $i$
   \[ \rightarrow d_{ij} \leq 3\alpha_j. \]

Directly connected to (temp open) $i'$
conflicts with $i$.
exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.
When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.
\[ \alpha_{j'} \text{ stopped no later (..maybe earlier..)} \]
\[ \alpha_j \leq \alpha_{j'}. \]
Total distance from $j$ to $i'$.
**Claim:** Client $j$ is indirectly connected to $i$

$\rightarrow d_{ij} \leq 3\alpha_j.$

Directly connected to (temp open) $i'$ conflicts with $i$.

exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.

$\alpha_{j'}$ stopped no later (..maybe earlier..)

$\alpha_j \leq \alpha_{j'}.$

Total distance from $j$ to $i'$.

$d_{ij} +$
Claim: Client $j$ is indirectly connected to $i$

$\implies d_{ij} \leq 3\alpha_j.$

Directly connected to (temp open) $i'$ conflicts with $i$.

exists $j'$ with $\alpha_{j'} \geq d_{i'j'}$ and $\alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.

$\alpha_{j'}$ stopped no later (..maybe earlier..)

$\alpha_j \leq \alpha_{j'}.$

Total distance from $j$ to $i'$.

$d_{ij} + d_{i'j'} +$
**Claim:** Client $j$ is indirectly connected to $i$
\[ \rightarrow d_{ij} \leq 3\alpha_j. \]

Directly connected to (temp open) $i'$ conflicts with $i$.
exists $j' \text{ with } \alpha_{j'} \geq d_{ij'} \text{ and } \alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_j'$.

$\alpha_j'$ stopped no later (..maybe earlier..)

$\alpha_j \leq \alpha_j'$.

Total distance from $j$ to $i'$.
\[ d_{ij} + d_{ij'} + d_{i'j'} \]
Claim: Client $j$ is indirectly connected to $i$
$\rightarrow d_{ij} \leq 3\alpha_j$

Directly connected to (temp open) $i'$$
conflicts with $i$.
exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.
When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.
$\alpha_{j'}$ stopped no later (..maybe earlier..)
$\alpha_j \leq \alpha_{j'}$.
Total distance from $j$ to $i'$.
$d_{ij} + d_{ij'} + d_{i'j'} \leq 3\alpha_j$
**Claim:** Client $j$ is indirectly connected to $i$

$\rightarrow d_{ij} \leq 3\alpha_j$.

Directly connected to (temp open) $i'$ conflicts with $i$.

- exists $j'$ with $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.

When $i'$ opens, stops both $\alpha_j$ and $\alpha_{j'}$.

- $\alpha_{j'}$ stopped no later (..maybe earlier..)
- $\alpha_j \leq \alpha_{j'}$.

Total distance from $j$ to $i'$.

$\quad d_{ij} + d_{ij'} + d_{i'j'} \leq 3\alpha_j$
Putting it together!

**Claim:** Client only pays one facility.

**Claim:** $S_i$ - directly connected clients to open facility $i$. 
Putting it together!

**Claim:** Client only pays one facility.

**Claim:** $S_i$ - directly connected clients to open facility $i$.

\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]

**Claim:** Client $j$ is indirectly connected to $i$

Total Cost: dual pays for facility and own connections. plus no more than 3 times indirect client dual.

Total Cost: 3 times dual.

feasible dual upper bounds fractional (and integer) primal.

3 OPT.

Fast!

Cheap!

Safe!
Putting it together!

**Claim:** Client only pays one facility.

**Claim:** $S_i$ - directly connected clients to open facility $i$.

$$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j.$$

**Claim:** Client $j$ is indirectly connected to $i$

$$\rightarrow d_{ij} \leq 3\alpha_j.$$
Claim: Client only pays one facility.
Claim: $S_i$ - directly connected clients to open facility $i$.

\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]

Claim: Client $j$ is indirectly connected to $i$

\[ \rightarrow d_{ij} \leq 3\alpha_j. \]

Total Cost:
Claim: Client only pays one facility.
Claim: $S_i$ - directly connected clients to open facility $i$.
\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]
Claim: Client $j$ is indirectly connected to $i$
\[ \rightarrow d_{ij} \leq 3\alpha_j. \]

Total Cost:
direct clients dual ($\alpha_j$) pays for facility and own connections.
Claim: Client only pays one facility.
Claim: $S_i$ - directly connected clients to open facility $i$.

$$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j.$$

Claim: Client $j$ is indirectly connected to $i$

$$\rightarrow d_{ij} \leq 3\alpha_j.$$

Total Cost:
direct clients dual ($\alpha_j$) pays for facility and own connections.
plus no more than 3 times indirect client dual.
Claim: Client only pays one facility.
Claim: $S_i$ - directly connected clients to open facility $i$.
\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]
Claim: Client $j$ is indirectly connected to $i$
\[ \rightarrow d_{ij} \leq 3\alpha_j. \]

Total Cost:
direct clients dual ($\alpha_j$) pays for facility and own connections.
plus no more than 3 times indirect client dual.
Total Cost: 3 times dual.
Putting it together!

Claim: Client only pays one facility.
Claim: $S_i$ - directly connected clients to open facility $i$.

$$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j.$$

Claim: Client $j$ is indirectly connected to $i$

$$\rightarrow d_{ij} \leq 3\alpha_j.$$

Total Cost:
direct clients dual ($\alpha_j$) pays for facility and own connections.
plus no more than 3 times indirect client dual.
Total Cost: 3 times dual.
feasible dual upper bounds fractional (and integer) primal.
Claim: Client only pays one facility.
Claim: $S_i$ - directly connected clients to open facility $i$.
\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_{j} \alpha_j. \]
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Putting it together!

**Claim:** Client only pays one facility.

**Claim:** $S_i$ - directly connected clients to open facility $i$.

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Total Cost:

direct clients dual ($\alpha_j$) pays for facility and own connections.

plus no more than 3 times indirect client dual.

Total Cost: 3 times dual.

feasible dual upper bounds fractional (and integer) primal.

3 OPT.
Claim: Client only pays one facility.

Claim: $S_i$ - directly connected clients to open facility $i$.

\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]

Claim: Client $j$ is indirectly connected to $i$

\[ \rightarrow d_{ij} \leq 3\alpha_j. \]

Total Cost:

direct clients dual ($\alpha_j$) pays for facility and own connections.
plus no more than 3 times indirect client dual.
Total Cost: 3 times dual.
feasible dual upper bounds fractional (and integer) primal.

3 OPT.

Fast!
Putting it together!

**Claim:** Client only pays one facility.

**Claim:** $S_i$ - directly connected clients to open facility $i$.

$$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j.$$

**Claim:** Client $j$ is indirectly connected to $i$

$$\rightarrow d_{ij} \leq 3\alpha_j.$$

Total Cost:

direct clients dual ($\alpha_j$) pays for facility and own connections.
plus no more than 3 times indirect client dual.

Total Cost: 3 times dual.

feasible dual upper bounds fractional (and integer) primal.

3 OPT.

Fast! Cheap!
Putting it together!

**Claim:** Client only pays one facility.

**Claim:** $S_i$ - directly connected clients to open facility $i$.
\[ f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j. \]

**Claim:** Client $j$ is indirectly connected to $i$
\[ \rightarrow d_{ij} \leq 3\alpha_j. \]

**Total Cost:**
direct clients dual ($\alpha_j$) pays for facility and own connections.
plus no more than 3 times indirect client dual.

**Total Cost:** 3 times dual.

feasible dual upper bounds fractional (and integer) primal.

3 OPT.

Fast! Cheap! Safe!
See you on Thursday.