Lecture 23
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# Lecture 23

## 1 Fast Max-Flow

Given undirected graph $G(V, E)$ where each edge has capacity 1, the objective is to find the maximum flow from $s$ to $t$, such that the flow on an edge does not exceed its capacity. The running time for the best known max flow algorithm until recently was $O(m^{3/2})$. We discuss the algorithm from [1] for finding $\epsilon$-approximate max-flows in time $\widetilde{O}(m^{4/3}/\epsilon)$.

### 1.1 Electrical Flows

**Finding electrical flows:** Let $L$ be the Laplacian for graph $G(V, E)$ where edge $e$ has weight $\frac{1}{R_e}$. The potential vector $\phi$ for the electric flow and the current vector $i$ whose $j$-th component is the outgoing current at vertex $j$ are related as follows,

$$L\phi = i \qquad\qquad L^+ i = \phi \tag{1}$$

The Laplacian linear system solver can be used to find the electrical flow for any $i$ in time $\widetilde{O}(m)$, by finding $\phi$ and then the current through each edge using Ohm's law.

**Energy minimization:** The energy dissipated in the network by flow $f_e$ is $\sum_e f_e^2 R_e$, as a function of $\phi$ the energy is $E(\phi) = \sum_{u\sim v} \frac{(\phi_u - \phi_v)^2}{R_{uv}}$. Consider the problem of finding the minimum energy flow if some vertex potentials are fixed. The partial derivative of the energy with respect to free variables must be 0,

$$\frac{\partial E(\phi)}{\partial \phi_u} = \sum_{v\sim u} \frac{2(\phi_u - \phi_v)}{R_{uv}} = 2i_u \tag{2}$$

The outflowing current at free vertices is 0 for the minimum energy flow, that is the Kirchoff current laws are satisfied. The electrical flow is therefore the unique energy minimizing flow.

### 1.2 Experts for max flow

**Binary search:** Suppose we have an algorithm that constructs a flow of value $f$ if one exists. The value of the max flow $F^*$ can be found in $\log m$ iterations by binary search and the max flow can be found by running the algorithm with input $F^*$.

**Flow game:** The max flow game is played between the edge and flow players, the edge player plays edge $e$, the flow player plays flow $f$ with value $F$ and the gain for the edge player is $f_e$.

Suppose the flow player can guarantee an average loss $(1 + \epsilon)$ and a maximum loss of $\rho$ against any strategy played by the edge player. This is formalized as an $(\epsilon, \rho)$ oracle,

$$\begin{aligned} \sum_e w_e f_e &\leq (1+\epsilon)\sum_e w_e \\ f_e &\leq \rho \\ |f| &= F \end{aligned} \tag{3}$$

Then the following algorithm converges to a $(1+O(\epsilon))$ approximate flow in $T = O(\frac{\rho \log n}{\epsilon^2})$ rounds.

> 1. The edge player follows the experts algorithm with initial weights $w(e) = 1$ and update rule $w_e^{t+1} = w_e^t(1 + \frac{\epsilon}{\rho} f_e^t)$.
> 2. The flow player plays the flow $f_i$ output by the $(\epsilon, \rho)$ oracle against weights $w_i$.

Claim 1

*The average flow $f^* = \frac{1}{T}\sum_{i \in T} f_i$ has value $F$ and satisfies all the capacity constraints on edges within a factor of $\epsilon$, for $T > \frac{\rho \ln n}{\epsilon^2}$.*

Proof: The oracle ensures that the gain for the edge player is at most $(1+\epsilon)$ in each round. The gain of the best expert in retrospect against the flow $f^*$ is $\max_e f^*(e)$. The analysis of the experts algorithm with gains yields,

$$(1+\epsilon)T \geq G \geq \max_e f^*(e)T(1-\epsilon) - \frac{\rho \ln m}{\epsilon} \tag{4}$$

The average flow $f^*$ at the end of the procedure therefore satisfies all the capacity constraints up to a factor of $1 + O(\epsilon)$. □

### 1.3 Implementing the oracle

The $(\epsilon, \rho)$ oracle can be implemented in time $\widetilde{O}(m)$ by finding electrical flows. The running time of the $\epsilon$ algorithm is $\widetilde{O}(\frac{m\rho}{\epsilon^2})$. We will first construct an oracle of width $\widetilde{O}(m^{1/2})$ and then improve the width to $\widetilde{O}(m^{1/3})$.

#### 1.3.1 A width $\sqrt{m}$ oracle

An oracle of width $\widetilde{O}(m^{1/2})$ can be implemented by finding an electrical flow of value $F$ from $s$ to $t$ with resistances as in the claim below,

Claim 2

*The electrical flow with resistances $R_e = w_e + \frac{\epsilon W}{m}$ has average congestion $(1+\epsilon)$ and maximum congestion $\widetilde{O}(m^{1/2})$.*

Proof: We know that there is a feasible flow $f'$ of value $F$ such that $f'(e) \leq 1$ for all $e$. The energy of the electrical flow is bounded by,

$$\sum_e f_e^2 R_e \quad \leq \quad \sum_e (f_e')^2 R_e \leq \sum_e R_e = (1+\epsilon)W$$

The average congestion can be bounded using the Cauchy Schwarz inequality,

$$\begin{aligned}\left(\sum_e w_e f_e\right)^2 &\leq \left(\sum_e w_e f_e^2\right)\left(\sum_e w_e\right) \\ &\leq \left(\sum_e R_e f_e^2\right) W \leq (1+\epsilon)W^2 \end{aligned} \tag{5}$$

Taking square roots, we have $\sum_e w_e f_e \leq \sqrt{1+\epsilon}W < (1+\epsilon)W$. To bound on the maximum congestion for the electrical flow, we note that the energy dissipated on an edge is at most the total energy of the electrical flow,

$$\frac{f_e^2 \epsilon W}{m} \leq (1+\epsilon)W \Rightarrow f_e \leq \sqrt{\frac{m(1+\epsilon)}{\epsilon}} \tag{6}$$

□

### 1.4 Improving the width of the oracle

The width of the oracle is improved by changing the algorithm to the following:

1. The edge player follows the experts algorithm with initial weights $w(e) = 1$ and update rule $w_e^{t+1} = w_e^t(1 + \frac{\epsilon}{\rho} f_e^t)$.
2. The flow player plays the flow $f_i$ outputs the electrical flow with $R_e = w_e + \frac{\epsilon W}{m}$, if there is an edge with congestion more than $\rho$ in $f_i$ delete edge and repeat.

The running time of this algorithm is $O(\frac{\rho m \log m}{\epsilon^2} + km)$, to analyze the algorithm we need to bound $k$ the number of edges removed and argue that the max flow does not change significantly due to removal of the edges.

### 1.5 Effective Resistance Lemma

The analysis relies on a lemma that bounds the change in effective resistance when an edge contributing a $\beta$ fraction of the energy of the electrical flow is removed.

Lemma 3
*Let $R(r)$ be the effective resistance between $(s,t)$ when edges have resistances $r_e$,*
*(i) If $r_e > r'_e$ then $R(r) \geq R(r')$.*
*(ii) Suppose $f$ is an electrical flow and $e$ an edge such that $f_e^2 r_e \geq \beta E_r(f)$. The effective resistance on removing $e$ is at least $\frac{R}{1-\beta}$.*

Proof: (i) For all potential vectors $\phi$ such that $\phi(s) = 1$ and $\phi(t) = 0$ the energy of the electrical flow corresponding to resistances $r$ is less than the energy for resistances $r'$.

$$E_r(\phi) = \sum_{u \sim v} \frac{(\phi_u - \phi_v)^2}{r_{uv}} \leq \sum_{u \sim v} \frac{(\phi_u - \phi_v)^2}{r'_{uv}} = E_{r'}(\phi) \tag{7}$$

Taking the minima we have $\frac{1}{R(r)} = \min_\phi E_r(\phi) \leq \min_\phi E_{r'}(\phi) = \frac{1}{R'(r)}$.

(ii) Let $\phi$ be the potential vector corresponding to the electrical flow with potential difference 1 across $s$ and $t$.

$$\frac{1}{R} \geq \sum_{e \in E(G) \setminus h} \frac{(\phi_u - \phi_v)^2}{r_e} + \frac{\beta}{R} \tag{8}$$

The energy of $\phi$ with respect to resistances $r'_e$ is at least $\frac{1}{R'}$, therefore $\frac{(1-\beta)}{R} \geq \frac{1}{R'}$.

(iii) Let us analyze the more general case when the resistance on an edge that carries a $\beta$ fraction of the energy is increased by $(1+\epsilon)$,

$$\frac{1-\beta}{R} + \frac{\beta}{R(1+\epsilon)} \geq \frac{1}{R'} \Rightarrow R' \geq R\frac{1+\epsilon}{1+\epsilon(1-\beta)} \geq (1+\frac{\epsilon\beta}{2})R$$

The last inequality is a more convenient form for later use, obtained by brute force.

□

## 1.6 Bounding the number of edges removed

THEOREM 4

*For $\rho = \widetilde{O}(m^{1/3})$, the number of edges removed by the algorithm is $\widetilde{O}(m^{1/3})$, the total capacity of the edges removed is $O(\epsilon F)$.*

PROOF: We analyze the change in the effective resistance between $(s,t)$ over the course of the algorithm. The effective resistance $R$ is the energy of the electrical flow of value 1.

The capacity of the $(s,t)$ min-cut is equal to $F^*$ by the max-flow min-cut theorem. There must be an edge with flow $\frac{1}{F^*}$ across the min-cut so the initial energy $R(0) \geq \frac{1}{(F^*)^2}$.

The energy dissipated of an edge with congestion $\rho$ is at least $\rho^2 R_e \geq \frac{\epsilon\rho^2 W}{m}$, while the total energy is at most $(1+\epsilon)W$. If $k$ edges get removed during the course of the algorithm, by part (ii) of the effective resistance lemma,

$$R(T) \geq R(0)\left(1 - \frac{\epsilon\rho^2}{m(1+\epsilon)}\right)^{-k}$$

The energy of the flow of value $F$ is at most $(1+\epsilon)W(T)$, hence we have an upper bound on $R(T)$,

$$\frac{(1+\epsilon)W(T)}{F^2} \geq R(T)$$

The increase in weight over a single iteration is bounded as follows,

$$W(t+1) = \sum_e w_e(t)\left(1 + \frac{\epsilon}{\rho}f_e^t\right) \leq W(t)\left(1 + \frac{\epsilon(1+\epsilon)}{\rho}\right) \tag{9}$$

The weight $W(T)$ after $T = \frac{\rho \ln n}{\epsilon^2}$ rounds can is at most $e^{2\ln n/\epsilon}$.

$$(1+\epsilon)e^{2\ln n/\epsilon} \geq \frac{F^2}{(F^*)^2}\left(1 - \frac{\epsilon\rho^2}{m(1+\epsilon)}\right)^{-k} \tag{10}$$

The ratio between $F^*$ and $F$ can be at most $m$, taking logarithms we have $k \leq \frac{2\ln m + 2\ln n/\epsilon}{-\ln(1 - \frac{\epsilon\rho^2}{m(1+\epsilon)})} = O\left(\frac{m\ln m}{\epsilon^2\rho^2}\right)$.

Choosing $\rho = O(m^{1/3}(\ln m)^{1/3}/\epsilon)$ we find that at most $O((m\ln m)^{1/3})$ edges get removed. The congestion $\rho$ on an edge can be at most $F$ [for the unit capacity case $F/\rho > 1$], so the total capacity of the edges removed is $\leq O\left(\frac{m\ln mF}{\epsilon^2\rho^3}\right) = O(\epsilon F)$. □

### 1.7 The Cut Algorithm

An $(s,t)$ cut can be found from potential vector $\phi$ such that $\phi(s) = 1$ and $\phi(t) = 0$ by choosing the minimum sweep cut. The expected value of a sweep cut obtained by choosing $t \in [0,1]$ uniformly at random is $\sum_{u,v} |\phi(u) - \phi(v)|$.

This can be bounded in terms of effective resistances using the Cauchy Schwartz inequality,

$$\sum_{e \in E} \phi(e) \leq \left( \left( \sum_e \frac{\phi_e^2}{r_e} \right) \left( \sum_e r_e \right) \right)^{1/2} = \sqrt{\frac{R}{R_{eff}}} \qquad (11)$$

Here is an algorithm that produces approximately minimum cuts,

1. Initialize weights $w_e(0) = 1$ for all edges, in iteration $t$ find electric flow with resistances $r(e) = w_e(t)$.
2. Update weights as $w_e(t+1) = w_e(t) + \frac{\epsilon}{\rho} f_e(t) + \frac{\epsilon^2}{m\rho} W(t)$.
3. If the minimum sweep cut has value less than $(1+6\epsilon)F$ output the min sweep cut.

Claim 5
*The algorithm produces a $(1 + O(\epsilon))$ min-cut in $N = O(m^{1/3} \log m)$ iterations with $\rho = O(m^{1/3})$.*

Proof: If we manage to show that within $N$ iterations the effective resistance $R_{eff} \geq \frac{(1-6\epsilon)W(t)}{F^2}$, the expected value of the sweep cut is at most $F(1 + O(\epsilon))$ by equation (11).

We will work under the assumption $R_{eff} \leq \frac{(1-6\epsilon)W(t)}{F^2}$. The total weight can be bounded as follows:

$$W(t+1) = W(t) \left( 1 + \frac{\epsilon^2}{\rho} \right) + \frac{\epsilon}{\rho} \sum w_e f_e \leq W(t) \left( 1 + \frac{\epsilon(1-2\epsilon)}{\rho} \right)$$

The average congestion $\sum w_e f_e \leq \sqrt{E(f)W} \leq \sqrt{1-6\epsilon} W \leq (1-3\epsilon)W$ the first inequality by Cauchy Schwartz and second by the assumption on effective resistance.

We want to argue that a large fraction of the weight is concentrated on edges in the min-cut, we introduce the following potential function, (Note that $\nu(t) \leq \max_{e \in e} w_e(t) < W(t)$).

$$\nu(t) = \left( \prod_{e \in C} w_e(t) \right)^{1/|C|}$$

For all rounds such that congestion is less than $\rho$, the change in $\nu(t)$ can be bounded as follows:

$$\nu(t+1) = \nu(t) \prod_{e \in C} \left( 1 + \frac{\epsilon f_e}{\rho} \right)^{1/|C|} \geq \nu(t) e^{\frac{\epsilon(1-\epsilon)}{\rho} \sum_{e \in C} \frac{f_e}{|C|}} \geq \nu(t) e^{\frac{\epsilon(1-\epsilon)}{\rho}}$$

We used the inequality $(1 + \epsilon x) \geq e^{x\epsilon(1-\epsilon)}$ for the second inequality and that $|F|/|C| > 0$ for the third.

The number of rounds $a$ for which the maximum congestion is less than $\rho$ can be bounded as follows,

$$e^{\frac{a\epsilon(1-\epsilon)}{\rho}} \leq \nu(T) \leq W(T) \leq me^{\frac{\epsilon(1-2\epsilon)T}{\rho}}$$

Taking logs and rearranging, $a \leq (1-\epsilon)T + \frac{\rho}{\epsilon}\log m$.

Now we need to bound $b$ the number of rounds where there is an edge with congestion more than $\rho$, for such rounds the effective resistance increases significantly,

$$\left(1 + \frac{\epsilon\beta}{2}\right)^b R(0) \leq (1 + O(\epsilon))me^{\frac{\epsilon(1-2\epsilon)T}{\rho}}$$

The energy fraction $\beta = \frac{\rho^2\epsilon}{m}$ and the initial resistance is $1/poly(m)$, taking logs and approximating (not accurate but ok essentially),

$$b \leq \frac{m}{\rho^2\epsilon^2}\left(\ln m + \frac{\epsilon T}{\rho}\right) = \widetilde{O}(\frac{mT}{\rho^3} + \frac{m}{\rho^2})$$

Th ebound on $a$ suggests choosing $T = \rho$ and the bound on $b$ suggests $T = \frac{m}{\rho^2}$ so $\rho = O(m^{1/3})$ is an optimal choice of parameters. □

## References

[1] P. Christiano, J.A. Kelner, A. Madry, D.A. Spielman, and S.H. Teng. Electrical Flows, Laplacian Systems, and Faster Approximation of Maximum Flow in Undirected Graphs. *Arxiv preprint arXiv:1010.2921*, 2010.