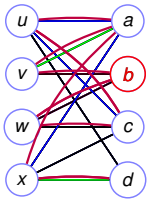
Today

Maximum Weight Matching
Undergraduate: saw maximum matching! (hopefully.) Will review.

Matching.

Given a bipartite graph, $G = (U, V, E)$, with edge weights $w : E \to R$, find a maximum weight matching.

A matching is a set of edges where no two share an endpoint.



Blue – 3. Green - 2,
Black - 1, Non-edges - 0.

Solution Value: 7.

Solution Value: 7.

Solution Value: 8.

Applications

Jobs to workers.

Teachers to classes.

Classes to classrooms.

"The assignment problem"

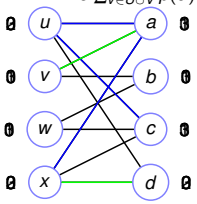
Min Weight Matching.
Negate values and find maximum weight matching.

Vertex Cover

Given a bipartite graph, $G = (U, V, E)$, with edge weights $w : E \to R$, find an vertex cover function of minimum total value.

A function $p : V \to R$, where for all edges, $e = (u, v)$, $p(u) + p(v) \geq w(e)$.

Minimize $\sum_{v \in U \cup V} p(u)$.



Solution Value: 12.

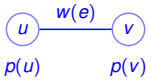
Solution Value: 12.

Solution Value: 9.

Solution Value: 8.

Cover is upper bound.

Feasible $p(\cdot)$, for edge $e = (u, v)$, $p(u) + p(v) \geq w(e)$.

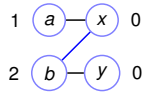


For a matching M, each u is the endpoint of at most one edge in M.

$$\sum_{e=(u,v) \in M} w(e) \leq \sum_{e=(u,v) \in M} (p(u) + p(v)) \leq \sum_{u \in U} p(u) + \sum_{v \in V} p(v)$$

Holds with equality if
for $e \in M$, $w(e) = p(u) + p(v)$ (Defn: tight edge.) and
perfect matching.

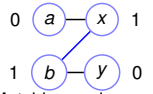
Simple example.



Blue edge – 2, Others – 1.

Using max incident edge.

Value: 3.
Using max incident edge.



Value: 2.
Same as optimal matching!

Proof of optimality.

Matching and cover are optimal,
edges in matching have $w(e) = p(u) + p(v)$. Tight edge.
all nodes are matched.

Maximum Matching

Given a bipartite graph, $G = (U, V, E)$, find a maximum sized matching.

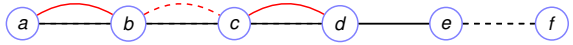
Size of maximum matching of a path on length n?
(A) $n/2$
(B) $\lfloor n/2 \rfloor$

(Rao would have said (A), don't worry.)

Why?
Alg: Start at end, and alternately put in edge or not.

What if one has a partial matching.

How do you make it bigger?



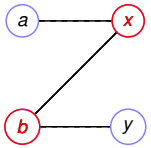
Greedily adding fails. So how? Augmenting Alternating Path. Switch!

Maximum Matching

Given a bipartite graph, $G = (U, V, E)$, find a maximum sized matching.

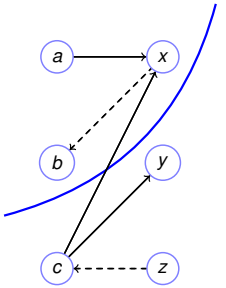
Key Idea: Augmenting Alternating Paths.

Example:



Start at unmatched node(s),
follow unmatched edge(s),
follow matched.
Repeat until an unmatched node.

No perfect matching



Can't increase matching size.
No alternating path from (a) to (y).

Cut!

Still no augmenting path.
Still Cut?

Use directed graph!
Cut in this graph.

Algorithm:
Given matching.
Direct unmatched edges U to V, matched V to U.
Find path between unmatched nodes on left to right. (BFS, DFS).
Until everything matched ... or output a cut.

Back to Maximum Weight Matching.

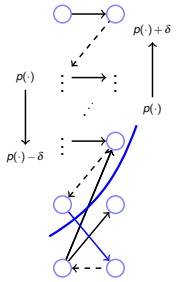
Want vertex cover (price function) $p(\cdot)$ and matching where.

Optimal solutions to *both* if
for $e \in M$, $w(e) = p(u) + p(v)$ (Defn: tight edge.) and perfect matching.

Maximum Weight Matching

Goal: perfect matching on tight edges.

Algorithm



Init: empty matching, feasible cover function ($p(\cdot)$)

Add tight edges to matching.
Use alt./aug. paths of tight edges.
"maximum matching algorithm."

No augmenting path.
Cut, (S, T), in directed graph of tight edges!

All edges across cut are not tight. (loose?)

Nontight edges leaving cut, go from S_U, T_V.

Lower prices in S_U, raise prices in S_V,
all explored edges still tight,
matched edges still tight
... and get new tight edge!
What's delta? $w(e) < p(u) + p(v) \rightarrow$
$\delta = min_{e \in (S_U \times T_V)} p(u) + p(v) - w(e)$.

Some details

Add 0 value edges, so that optimal solution contains perfect matching.

Beginning "Matcher" Solution: $M = \{\}$.

Feasible! Value = 0.

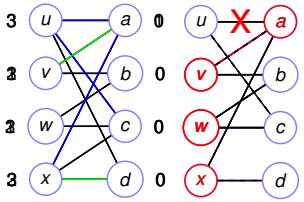
Beginning "Coverer" Solution:
$p(u) =$ maximum incident edge for $u \in U$, 0 otherwise.

Main Work:
breadth first search from unmatched nodes finds cut.
Update prices (find minimum delta.)

Simple Implementation:
Each bfs either augments or adds node to S in next cut.
$O(n)$ iterations per augmentation.
$O(n)$ augmentations.

$O(n^2 m)$ time.

Example



Weight legend:
black 1, green 2, blue 3
Tight edges for inital prices.
Max matching in tight edges.
dashed means matched.
No augmenting path →
reachable: $S = \{u, w\}$
Blue edge on right soon
to be tight!
Adjust prices...$\delta = 1$
new tight edges.
Still no augmenting path.
Reachable $S = \{v, w, x, a\}$
Blue edges minimally non-tight.
Adjust prices.
Some more tight edges.
And X shows
a "new" nontight edge.
..and another augmentation...

..and finally: a perfect matching.

All matched edges tight.
Perfect matching. Feasible price function. Values the same. Optimal!

Notice:
no weights on the right problem.
retain previous matching through price changes.
retains edges in failed search through price changes.

Perfect Matching

Linear program: $\max \sum_e w_e x_e$, $\forall v : \sum_{e=(u,v)} x_e \leq 1$, $x_e \geq 0$

Dual: $\min \sum_v p_v$, $\forall e = (u, v) : p_u + p_v \geq w_e$, $p_u \geq 0$.

In this case:
Dual feasible at start: $p_u \geq \max_{e=(u,v)} w_e$
Maintain feasibility: adjust prices by δ.
Maintain Primal feasibility.
Maintain complementary slackness (2).
$x_e > 0$ only if $p_u + p_v = w_e$.
Eventually match all vertices.

The Engine that pulls the train: Find a path of tight edges.

Complementary slackness (1): Terminate when perfect matching.
$\forall v : \sum_{e=(u,v)} x_e$. So any p_u can be non-zero.

The "play" indicates game playing.
Algorithm plays lower bound against upper bound.
Two person games: von Neuman. Equilibrium: Nash.

Is the path fundamental?
Are things as easy or as hard as $0, 1, 2, \ldots\ldots$?

Linear Program.

How? From lecture warmup.

Linear program: $\max cx, Ax \leq b, x \geq 0$
Dual: $\min y^T b, y^T A \geq c, y \geq 0$

Note: Dual variables correspond to primal equations and vice versa.

Weak Duality:

$$y^T b \geq y^T A x \geq cx$$

First inequality from $b \geq Ax$ and second from $y^A \geq c$.

Complementary slackness:
(1) $x_j > 0 \implies a^{(j)} y = c_j$
(2) $y_i > 0 \implies a_i x = b_i$

What does multiplying by 0 do?
Zero and one. My love is won. Nothing and nothing done.

(2) $\implies y^T b = \sum_i y_i b_i = \sum_i y_i (a_i x) = y^T A x$.

Similarly: (1) $\implies y^T A x = cx$.

Complementary slackness conditions imply optimality.

...see you on Tuesday

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