Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.

Problem: Route path for each pair and minimize maximum congestion.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \dots (s_k, t_k)$.

Problem: Route path for each pair and minimize maximum congestion.

Congestion is maximum number of paths that use any edge.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \dots (s_k, t_k)$.

Problem: Route path for each pair and minimize maximum congestion.

Congestion is maximum number of paths that use any edge.

Note: Number of paths is exponential.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.

Problem: Route path for each pair and minimize maximum congestion.

Congestion is maximum number of paths that use any edge.

Note: Number of paths is exponential.
Can encode in polysized linear program, but large.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.

Problem: Route path for each pair and minimize maximum congestion.

Congestion is maximum number of paths that use any edge.

Note: Number of paths is exponential.
Can encode in polysized linear program, but large.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \dots (s_k, t_k)$.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \dots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max congestion on any edge.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max congestion on any edge.

Two person game.

Row is router.

Two person game.

Row is router.

An exponential number of rows!

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e, r]$ - congestion of edge e on routing r.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e, r]$ - congestion of edge e on routing r.

m rows.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e, r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e,r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Multiplicative Weights only maintains m weights.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e,r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Multiplicative Weights only maintains m weights.

Adversary only needs to provide best column each day.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e,r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Multiplicative Weights only maintains m weights.

Adversary only needs to provide best column each day.

Runtime only dependent on m and T (number of days.)

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}.$$

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}.$$

Let $T = \frac{k \log n}{\varepsilon^2}$

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:
$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T}$

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

$G^* = T * c_{\max}$ – Best row payoff against average routing (times T).

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

$G^* = T * c_{\max}$ – Best row payoff against average routing (times T).

$G \leq T \times C^*$ – each day, gain is avg. congestion $\leq$ opt congestion.

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

$G^* = T * c_{\max}$ – Best row payoff against average routing (times T).

$G \leq T \times C^*$ – each day, gain is avg. congestion $\leq$ opt congestion.

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:

$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

$G^* = T * c_{\max}$ – Best row payoff against average routing (times T).

$G \leq T \times C^*$ – each day, gain is avg. congestion $\leq$ opt congestion.

$T = \frac{k \log n}{\varepsilon^2} \rightarrow Tc^*_{\max} - TC \leq \varepsilon TC^* + \frac{k \log n}{\varepsilon} \quad \rightarrow$

$c_{max} - C^* \leq \varepsilon C^* + \varepsilon$ □

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

$O(\frac{k \log n}{\varepsilon^2})$ steps

Better setup.

Runtime: $O(km\log n)$ to route in each step (using Dijkstra's)

$O(\frac{k\log n}{\varepsilon^2})$ steps
to get $c_{\max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

$O(\frac{k \log n}{\varepsilon^2})$ steps
to get $c_{\max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

$O(\frac{k \log n}{\varepsilon^2})$ steps
to get $c_{\text{max}} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

To get constant c error.
$\rightarrow O(k^2 m \log n / \varepsilon^2)$ to get a constant approximation.

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

$O(\frac{k \log n}{\varepsilon^2})$ steps
to get $c_{\max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

To get constant c error.
$\rightarrow O(k^2 m \log n/\varepsilon^2)$ to get a constant approximation.
Exercise: $O(km \log n/\varepsilon^2)$ algorithm

Better setup.

Runtime: $O(km\log n)$ to route in each step (using Dijkstra's)

$O(\frac{k\log n}{\varepsilon^2})$ steps
to get $c_{\max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

To get constant c error.
$\rightarrow O(k^2 m\log n/\varepsilon^2)$ to get a constant approximation.
Exercise: $O(km\log n/\varepsilon^2)$ algorithm !

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

$O(\frac{k \log n}{\varepsilon^2})$ steps
to get $c_{\max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

To get constant c error.
$\rightarrow O(k^2 m \log n / \varepsilon^2)$ to get a constant approximation.
Exercise: $O(km \log n / \varepsilon^2)$ algorithm ! !

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

$O(\frac{k \log n}{\varepsilon^2})$ steps
to get $c_{\text{max}} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

To get constant c error.
$\rightarrow O(k^2 m \log n / \varepsilon^2)$ to get a constant approximation.
Exercise: $O(km \log n / \varepsilon^2)$ algorithm ! ! !

Fractional versus Integer.

Did we (approximately) solve path routing?

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes?

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No!

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
  We approximately solved fractional routing problem.

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
 We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)
$c(e)$ is relatively large

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)
$c(e)$ is relatively large ($\Omega(\log n)$)

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)
$c(e)$ is relatively large ($\Omega(\log n)$)
$\rightarrow \tilde{c}(e) \approx c(e)$.

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)
$c(e)$ is relatively large $(\Omega(\log n))$
$\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results?

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)
$c(e)$ is relatively large $(\Omega(\log n))$
$\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results? later.

Learning

Learning just a bit.

Learning

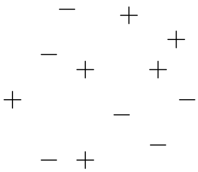
Learning just a bit.

Example: set of labelled points, find hyperplane that separates.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.

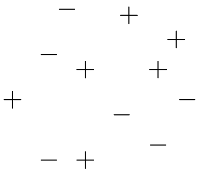


Looks hard.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



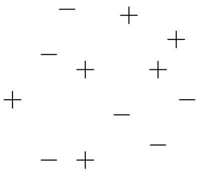
Looks hard.

1/2 of them?

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



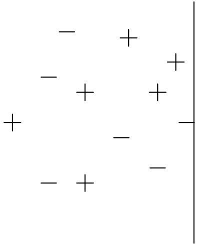
Looks hard.

$1/2$ of them? Easy.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



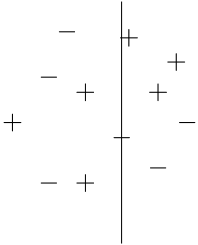
Looks hard.

$1/2$ of them? Easy.
Arbitrary line.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



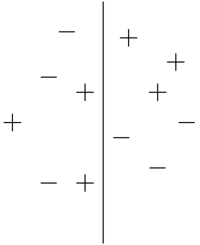
Looks hard.

$1/2$ of them? Easy.
Arbitrary line. And Scan.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



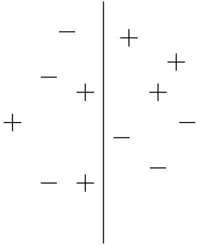
Looks hard.

1/2 of them? Easy.
Arbitrary line. And Scan.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

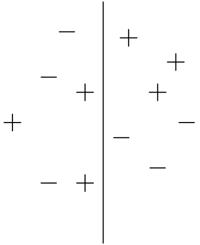
$1/2$ of them? Easy.
Arbitrary line. And Scan.

Useless.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

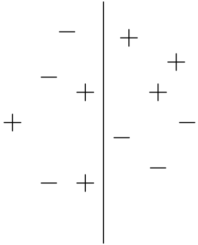
$1/2$ of them? Easy.
Arbitrary line. And Scan.

Useless. A bit more than $1/2$ Correct would be better.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

$1/2$ of them? Easy.
Arbitrary line. And Scan.

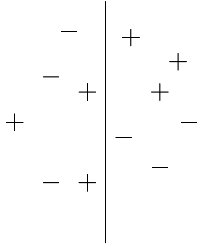
Useless. A bit more than $1/2$ Correct would be better.

Weak Learner: Classify $\geq \frac{1}{2} + \varepsilon$ points correctly.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

$1/2$ of them? Easy.
Arbitrary line. And Scan.

Useless. A bit more than $1/2$ Correct would be better.

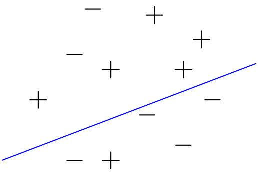
Weak Learner: Classify $\geq \frac{1}{2} + \varepsilon$ points correctly.

Not really important but ...

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

1/2 of them? Easy.
Arbitrary line. And Scan.

Useless. A bit more than 1/2 Correct would be better.

Weak Learner: Classify $\geq \frac{1}{2}+\varepsilon$ points correctly.

Not really important but ...

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:

produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Strong Learner:

Weak Learner/Strong Learner

Input: *n* labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2}+\varepsilon$ fraction

Strong Learner:
produce hyp. correctly classifies $1+\mu$ fraction

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Strong Learner:
produce hyp. correctly classifies $1 + \mu$ fraction
That's a really strong learner!

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:

produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Strong Learner:

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Strong Learner:
produce hypothesis correctly classifies $1 - \mu$ fraction

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2}+\varepsilon$ fraction

Strong Learner:
produce hypothesis correctly classifies $1-\mu$ fraction

Same thing?

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Strong Learner:
produce hypothesis correctly classifies $1 - \mu$ fraction

Same thing?

Can one use weak learning to produce strong learner?

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2} + \varepsilon$ fraction

Strong Learner:
produce hypothesis correctly classifies $1 - \mu$ fraction

Same thing?

Can one use weak learning to produce strong learner?

Boosting: use a weak learner to produce strong learner.

Poll.

Given a weak learning method (produce ok hypotheses.)

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes

(B) No

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes

(B) No

If yes.

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes

(B) No

If yes. How?

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes
(B) No

If yes. How?

The idea: Multiplicative Weights.

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes

(B) No

If yes. How?

The idea: Multiplicative Weights.
Standard online optimization method reinvented in many areas.

Boosting/MW Framework

Boosting/MW Framework

Points lose when classified correctly.

Boosting/MW Framework

Points lose when classified correctly.

The little devils want to fool the learner.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Boosting/MW Framework

Points lose when classified correctly.

The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

Boosting/MW Framework

Points lose when classified correctly.

The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Output hypotheses $h(x)$:

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by ($1 - \varepsilon$).

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by ($1 - \varepsilon$).

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points !

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points ! !

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points ! ! !

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by ($1 - \varepsilon$).

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points ! ! !

Cool!

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by $(1 - \varepsilon)$.

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points ! ! !

Cool!

Really?

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2} \ln \frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2 + \gamma$ of weighted points.
2. Multiply each point that is correct by ($1 - \varepsilon$).

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1 - \mu$ of the points ! ! !

Cool!

Really? Proof?

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \leq 1/2$, that $-x - x^2 \leq \ln(1-x) \leq -x$.

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \leq 1/2$, that $-x - x^2 \leq \ln(1-x) \leq -x$.

The first inequality is from geometric series.

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \leq 1/2$, that $-x - x^2 \leq \ln(1-x) \leq -x$.

The first inequality is from geometric series.

$x^3/3 + ... = x^2(x/3 + x^2/4 + ..)$

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \le 1/2$, that $-x - x^2 \le \ln(1-x) \le -x$.

The first inequality is from geometric series.

$x^3/3 + ... = x^2(x/3 + x^2/4 + ..) \le x^2(1/2)$ for $|x| < 1/2$.

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \leq 1/2$, that $-x - x^2 \leq \ln(1-x) \leq -x$.

The first inequality is from geometric series.

$x^3/3 + ... = x^2(x/3 + x^2/4 + ..) \leq x^2(1/2)$ for $|x| < 1/2$.

The second is from truncation.

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \leq 1/2$, that $-x - x^2 \leq \ln(1-x) \leq -x$.

The first inequality is from geometric series.

$x^3/3 + ... = x^2(x/3 + x^2/4 + ..) \leq x^2(1/2)$ for $|x| < 1/2$.

The second is from truncation.

Second implies: $(1-\varepsilon)^x \leq e^{-\varepsilon x}$, by exponentiation.

Adaboost proof.

Claim: $h(x)$ is correct on $1 - \mu$ of the points!

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

Adaboost proof.

Claim: $h(x)$ is correct on $1 - \mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning

Adaboost proof.

Claim: $h(x)$ is correct on $1 - \mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$$W(T) \geq (1-\varepsilon)^{\frac{T}{2}} |S_{bad}|$$

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}} |S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Loss $L_t \geq (1/2+\gamma)$

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Loss $L_t \geq (1/2+\gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Loss $L_t \geq (1/2+\gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow$

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}} |S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2} + \gamma$ of the weight.

Loss $L_t \geq (1/2 + \gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow W(T) \leq ne^{-\varepsilon \sum_t L_t}$

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}} |S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2} + \gamma$ of the weight.

Loss $L_t \geq (1/2 + \gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow W(T) \leq ne^{-\varepsilon \sum_t L_t} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Loss $L_t \geq (1/2+\gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow W(T) \leq ne^{-\varepsilon \sum_t L_t} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Combining

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$$

Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Loss $L_t \geq (1/2+\gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow W(T) \leq ne^{-\varepsilon \sum_t L_t} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Combining

$|S_{bad}|(1-\varepsilon)^{T/2} \leq W(T) \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Calculation..

$$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

Calculation..

$$|S_{bad}|(1-\varepsilon)^{T/2} \le ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \le -\gamma T(\frac{1}{2}+\gamma)$$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \le ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \le -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \le \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma - \gamma^2) \le -\gamma T(\frac{1}{2}+\gamma)$$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma - \gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \to \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

Calculation..

$$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \to \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

Calculation..

$$|S_{bad}|(1-\varepsilon)^{T/2} \le ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \le -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \le \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \le -\gamma T(\frac{1}{2}+\gamma) \rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \le -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$$\rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \le \log\mu$$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\tfrac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\tfrac{1}{2}+\gamma) \to \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\to \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \to \frac{|S_{bad}|}{n} \leq \mu.$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma - \gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \to \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\to \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \to \frac{|S_{bad}|}{n} \leq \mu.$

The misclassified set is at most μ fraction of all the points.

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \to \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\to \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \to \frac{|S_{bad}|}{n} \leq \mu.$

The misclassified set is at most μ fraction of all the points.

The hypothesis correctly classifies $1-\mu$ of the points

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \rightarrow \frac{|S_{bad}|}{n} \leq \mu.$

The misclassified set is at most μ fraction of all the points.

The hypothesis correctly classifies $1-\mu$ of the points !

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \rightarrow \frac{|S_{bad}|}{n} \leq \mu.$

The misclassified set is at most μ fraction of all the points.

The hypothesis correctly classifies $1-\mu$ of the points !

Claim: Multiplicative weights: $h(x)$ is correct on $1-\mu$ of the points!

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \le ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \le -\gamma T(\tfrac{1}{2}+\gamma)$$

Again, $-\gamma - \gamma^2 \le \ln(1-\gamma)$,

$$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \le -\gamma T(\tfrac{1}{2}+\gamma) \rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \le -\frac{\gamma^2 T}{2}$$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \le \log\mu \rightarrow \frac{|S_{bad}|}{n} \le \mu.$

The misclassified set is at most μ fraction of all the points.

The hypothesis correctly classifies $1-\mu$ of the points !

Claim: Multiplicative weights: $h(x)$ is correct on $1-\mu$ of the points!

Some details...

Weak learner learns over distributions of points not points.

Some details...

Weak learner learns over distributions of points not points.

- Make copies of points to simulate distributions.

Some details...

Weak learner learns over distributions of points not points.

- Make copies of points to simulate distributions.

Used often in machine learning.

Some details...

Weak learner learns over distributions of points not points.

Make copies of points to simulate distributions.

Used often in machine learning.

Blending learning methods.

Theme:Good on average, hyperplane.

“Duality”

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$
 Find a solution for just one constraint!!!
 Best response.

Multiplicative weights: two person games (linear programs)
 y is exponential weights on "how unsatisfied" each equation is.
 $y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$
 y "wins" $\equiv$ unsatisfiable linear combo of constraints.
 Otherwise, x eventually "wins".

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Or doesn't and x responds.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Or doesn't and x responds.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0.$

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Or doesn't and x responds.

The math:

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Or doesn't and x responds.

The math: e

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Or doesn't and x responds.

The math: $e =$

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$

Find a solution for just one constraint!!!

Best response.

Multiplicative weights: two person games (linear programs)

y is exponential weights on "how unsatisfied" each equation is.

$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$

y "wins" $\equiv$ unsatisfiable linear combo of constraints.

Otherwise, x eventually "wins".

Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?

y^T "separates" affine subspace Ax from $\geq y^T c$.

Or doesn't and x responds.

The math: $e = \lim_{n\to\infty}(1+1/n)^n$.

Theme:Good on average, hyperplane.

"Duality"

$\min cx, Ax \geq b, x \geq 0$.

Linear combination of constraints: $y^T Ax \geq y^T c$
Find a solution for just one constraint!!!
Best response.

Multiplicative weights: two person games (linear programs)
y is exponential weights on "how unsatisfied" each equation is.
$y_i \propto \sum_t (1+\varepsilon)^{(a_i x^{(t)} - b_i)}$
y "wins" $\equiv$ unsatisfiable linear combo of constraints.
Otherwise, x eventually "wins".
Or pair that are pretty close.

(Apologies: switched x and y in game setup.)

"Separating" Hyperplane?
y^T "separates" affine subspace Ax from $\geq y^T c$.
Or doesn't and x responds.

The math: $e = \lim_{n \to \infty} (1 + 1/n)^n$.

A step closer.

Another Algorithm.

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.

If $x^{(t)}$ point violates constraint by $> \varepsilon$

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.
If $x^{(t)}$ point violates constraint by $> \varepsilon$
move toward constraint.

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.
If $x^{(t)}$ point violates constraint by $> \varepsilon$
move toward constraint.
Closer.

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.
If $x^{(t)}$ point violates constraint by $> \varepsilon$
move toward constraint.
Closer.

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.
If $x^{(t)}$ point violates constraint by $> \varepsilon$
move toward constraint.
Closer.

The Math:

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.
If $x^{(t)}$ point violates constraint by $> \varepsilon$
move toward constraint.
Closer.

The Math:
Wrong side, angle to correct point is less than 90°

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.
If $x^{(t)}$ point violates constraint by $> \varepsilon$
move toward constraint.
Closer.

The Math:
Wrong side, angle to correct point is less than 90°

This is the idea in perceptron.

A step closer.

Another Algorithm.

Finding a feasible point: x^* for constraints.

If $x^{(t)}$ point violates constraint by $> \varepsilon$

move toward constraint.

Closer.

The Math:

Wrong side, angle to correct point is less than 90°

This is the idea in perceptron. But can do analysis directly.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

The math:

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

The math:
linear (and quadratic) approximation of e^x.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

The math:
linear (and quadratic) approximation of e^x.
Advantage?

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

The math:
linear (and quadratic) approximation of e^x.
Advantage?
Distributions have entropy at most $O(\log n)$.

Multiplicative weights and a step closer.

The solution is a distribution: p^*.

Every day each strategy loses (or not), $\ell_i^{(t)}$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance (divergence) from q to p $\sum_i p_i^* \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
so new distribution plays losers less.
Move toward playing losers less.
Thus closer to p^*.

The math:
linear (and quadratic) approximation of e^x.
Advantage?
Distributions have entropy at most $O(\log n)$.

Reinforcement learning == Bandits.

Multiplicative Weights framework:

Reinforcement learning == Bandits.

Multiplicative Weights framework:

Update all experts.

Reinforcement learning == Bandits.

Multiplicative Weights framework:

Update all experts.

Reinforcement learning == Bandits.

Multiplicative Weights framework:

Update all experts.

Bandits.

Reinforcement learning == Bandits.

Multiplicative Weights framework:

Update all experts.

Bandits.

Only update expert you choose.

Reinforcement learning == Bandits.

Multiplicative Weights framework:

Update all experts.

Bandits.

Only update expert you choose.

No information about others.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
    No information about others.
(Named after one-armed bandit slot machine.)

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
Update all experts.

Bandits.
Only update expert you choose.
No information about others.
(Named after one-armed bandit slot machine.)

Idea: "Learn" which expert is best.
Prof. Dragan's mantra: formulation as optimization.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
   Prof. Dragan’s mantra: formulation as optimization.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
   Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
   Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
Update all experts.

Bandits.
Only update expert you choose.
No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: "Learn" which expert is best.
   Prof. Dragan's mantra: formulation as optimization.

Exploration: choose new bandit to get "data".
Exploitation: choose best bandit.

Strategy:

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: "Learn" which expert is best.
  Prof. Dragan's mantra: formulation as optimization.

Exploration: choose new bandit to get "data".
Exploitation: choose best bandit.

Strategy:
  Multiplicative weights.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
   Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Strategy:
   Multiplicative weights.
   Update by $(1+\varepsilon)$.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: "Learn" which expert is best.
  Prof. Dragan's mantra: formulation as optimization.

Exploration: choose new bandit to get "data".
Exploitation: choose best bandit.

Strategy:
  Multiplicative weights.
  Update by $(1+\varepsilon)$.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
  Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Strategy:
  Multiplicative weights.
  Update by $(1+\varepsilon)$.

Big ε.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: "Learn" which expert is best.
   Prof. Dragan's mantra: formulation as optimization.

Exploration: choose new bandit to get "data".
Exploitation: choose best bandit.

Strategy:
   Multiplicative weights.
   Update by $(1+\varepsilon)$.

Big ε.
  Exploit or explore more?

Reinforcement learning == Bandits.

Multiplicative Weights framework:
Update all experts.

Bandits.
Only update expert you choose.
No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Strategy:
Multiplicative weights.
Update by $(1+\varepsilon)$.

Big ε.
Exploit or explore more? Exploit.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
    No information about others.
(Named after one-armed bandit slot machine.)

Idea: "Learn" which expert is best.
  Prof. Dragan's mantra: formulation as optimization.

Exploration: choose new bandit to get "data".
Exploitation: choose best bandit.

Strategy:
  Multiplicative weights.
  Update by $(1+\varepsilon)$.

Big ε.
  Exploit or explore more? Exploit.

Perceptron also like bandits.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
Update all experts.

Bandits.
Only update expert you choose.
No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Strategy:
Multiplicative weights.
Update by $(1+\varepsilon)$.

Big ε.
Exploit or explore more? Exploit.

Perceptron also like bandits.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
   No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
   Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Strategy:
   Multiplicative weights.
   Update by $(1+\varepsilon)$.

Big ε.
  Exploit or explore more? Exploit.

Perceptron also like bandits. One point at a time.

Reinforcement learning == Bandits.

Multiplicative Weights framework:
  Update all experts.

Bandits.
  Only update expert you choose.
    No information about others.
(Named after one-armed bandit slot machine.)

Idea: “Learn” which expert is best.
  Prof. Dragan’s mantra: formulation as optimization.

Exploration: choose new bandit to get “data”.
Exploitation: choose best bandit.

Strategy:
  Multiplicative weights.
  Update by $(1+\varepsilon)$.

Big ε.
  Exploit or explore more? Exploit.

Perceptron also like bandits. One point at a time.

Online optimization: limited information.

Next up: convex optimization.

Analysis of previous.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

Idea: infeasible gives direction to step toward a feasible point.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

Idea: infeasible gives direction to step toward a feasible point.
violation of hyperplane for perceptron.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

- Idea: infeasible gives direction to step toward a feasible point.
  - violation of hyperplane for perceptron.
  - loss function for multiplicative weights.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

- Idea: infeasible gives direction to step toward a feasible point.
  - violation of hyperplane for perceptron.
  - loss function for multiplicative weights.

Next up: convex optimization.

Analysis of previous.
Get closer to a feasible point.

Idea: infeasible gives direction to step toward a feasible point.
violation of hyperplane for perceptron.
loss function for multiplicative weights.

Next: Get closer to an optimal point for function.

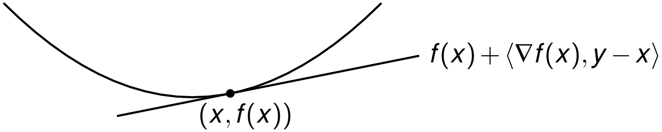
Convex optimization

Slides: Thanks to Di Wang.

$$\min_{x \in Q} f(x)$$

$$f(x) - f(y) \leq \langle \nabla f(x), x - y \rangle$$

Q: feasible space, convex.



Convex optimization

Slides: Thanks to Di Wang.

$$\min_{x \in Q} f(x)$$

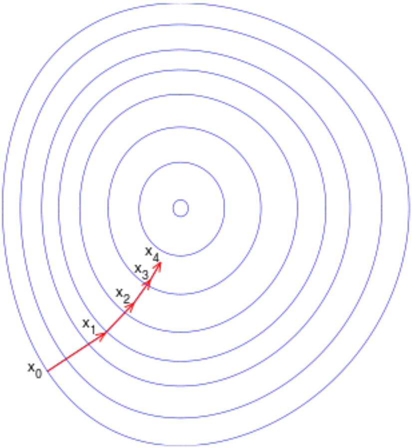
$$f(x) - f(y) \leq \langle \nabla f(x), x - y \rangle$$

Q: feasible space, convex.

First-order Iterative Methods

- Query $x \in Q$, update using $\nabla f(x)$
- Low per-iteration cost, $\mathrm{poly}(\frac{1}{\varepsilon})$ convergence.
- Methods of choice in large-scale regime.

Gradient Descent



- Moves in down-hill direction.
- Improve objective function value each iteration.
- Output final point.

Gradient Descent

L-Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x - y\| \quad \forall x, y \in Q$$

- Global linear lower bound and quadratic upper bound:

$$\forall y \quad f(x) + \langle \nabla f(x), y - x\rangle \leq f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2$$

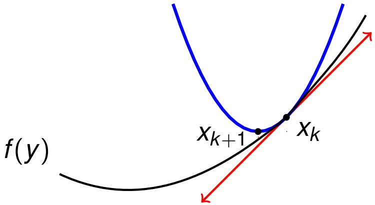
Gradient Descent

L-Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x - y\| \quad \forall x, y \in Q$$

- Global linear lower bound and quadratic upper bound:

$$\forall y \quad f(x) + \langle \nabla f(x), y - x \rangle \leq f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2$$



Gradient Descent

L-Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x - y\| \quad \forall x, y \in Q$$

- Global linear lower bound and quadratic upper bound:

$$\forall y \quad f(x) + \langle \nabla f(x), y - x \rangle \leq f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2$$

- Minimize using quadratic bound

$$x_{k+1} = \mathrm{Grad}(x_k) = \operatorname*{argmin}_{x \in Q} \{\langle \nabla f(x_k), x - x_k \rangle + \frac{L}{2}\|x - x_k\|^2\}$$

If $Q = \mathbb{R}^n$ and ℓ_2-norm, $x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k)$.

Gradient Descent

L-Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x - y\| \quad \forall x, y \in Q$$

- Global linear lower bound and quadratic upper bound:

$$\forall y \quad f(x) + \langle \nabla f(x), y - x \rangle \leq f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2$$

- Minimize using quadratic bound

$$x_{k+1} = \mathrm{Grad}(x_k) = \operatorname*{argmin}_{x \in Q}\{\langle \nabla f(x_k), x - x_k \rangle + \frac{L}{2}\|x - x_k\|^2\}$$

If $Q = \mathbb{R}^n$ and ℓ_2-norm, $x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k)$.

- **Primal progress**: Av. $\nabla f(x') \geq \frac{\nabla f(x)}{2}$ for $x' = \alpha x_k + (1 - \alpha)x_{k+1}$

$$f(x_k) - f(x_{k+1}) \geq \frac{1}{2L}\|\nabla f(x_k)\|_*^2$$

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$.

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Gradient Descent: one dimensional intuition.

Convexity:
$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*)$

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

Gradient Descent: one dimensional intuition.

Convexity:
$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T(x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:
$x^+ = x - \frac{1}{L}\nabla f(x)$

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x)$ $\qquad$ $f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR.

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T(x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR. Progress/step: Roughly $g^2/2$.

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x)$ $\qquad$ $f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR. Progress/step: Roughly $g^2/2$.

Idea: Gap/(progress/step) $\implies$ roughly $2LR/g$ steps.

Gradient Descent: one dimensional intuition.

Convexity:
$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) = gR$

L-Lipschitz, $R = \|x_0 - x^*\|$:
$x^+ = x - \frac{1}{L} \nabla f(x)$ $\quad$ $f(x) - f(x^+) \geq \frac{1}{2L} \|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR. Progress/step: Roughly $g^2/2$.

Idea: Gap/(progress/step) $\implies$ roughly $2LR/g$ steps.

Convexity: $g \geq (f(x) - f(x^*))/R$

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) = gR$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR. Progress/step: Roughly $g^2/2$.

Idea: Gap/(progress/step) $\implies$ roughly $2LR/g$ steps.

Convexity: $g \geq (f(x) - f(x^*))/R \implies 2LR^2/(f(x) - f(x^*))$ steps.

Gradient Descent: one dimensional intuition.

Convexity:
$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T(x - x^*) = gR$

L-Lipschitz, $R = \|x_0 - x^*\|$:
$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR. Progress/step: Roughly $g^2/2$.

Idea: Gap/(progress/step) $\implies$ roughly $2LR/g$ steps.

Convexity: $g \geq (f(x) - f(x^*))/R \implies 2LR^2/(f(x) - f(x^*))$ steps.

While gap $f(x) - f(x^*) \geq \varepsilon$ we have $g \geq \varepsilon/R$.

Gradient Descent: one dimensional intuition.

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

Also: $f(x) - f(x^*) \leq \nabla f(x)^T(x - x^*) = gR$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

In one dimension: $\nabla f(x) = g$.

Gap: gR. Progress/step: Roughly $g^2/2$.

Idea: Gap/(progress/step) $\implies$ roughly $2LR/g$ steps.

Convexity: $g \geq (f(x) - f(x^*))/R \implies 2LR^2/(f(x) - f(x^*))$ steps.

While gap $f(x) - f(x^*) \geq \varepsilon$ we have $g \geq \varepsilon/R$.

$\implies O(LR^2/\varepsilon)$ steps reduce gap by $1/2$.

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$.

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x).$ $\implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x)$

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

Gradient Descent: convergence in ℓ_2

Convexity:
$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:
$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}(\|x - x^*\|_2^2 -$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}(\|x - x^*\|_2^2 -$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2\right)$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2\right)$$

$$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}\left(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2\right)$$

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2\right)$$

$$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}\left(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T(x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T(x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2\right)$$

$$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}\left(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$$

Gradient Descent: convergence in ℓ_2

Convexity:

$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$

$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$

$\implies f(x^+) - f(x^*) \leq \frac{L}{2}(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2)$

$\leq \frac{L}{2}(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2)$ Add 0

$\leq \frac{L}{2}(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2)$
$\leq \frac{L}{2}(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2)$

$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2)$
$\leq \frac{L}{2}(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$

$f(x_k)$ is decreasing, we have $f(x_T) \leq \frac{1}{T}\sum_k f(x_k)$.

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2\right)$$

$$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}\left(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$$

$f(x_k)$ is decreasing, we have $f(x_T) \leq \frac{1}{T}\sum_k f(x_k)$.

$$\implies f(x_T) - f(x^*) \leq \frac{LR^2}{2T} \text{ where } R = \|x_0 - x^*\|.$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T (x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{1}{2L}\|\nabla f(x)\|_2^2$$

$$\implies f(x^+) - f(x^*) \leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2\right)$$

$$\leq \frac{L}{2}\left(\frac{2}{L}\nabla f(x)^T (x - x^*) - \frac{1}{L^2}\|\nabla f(x)\|_2^2 - \|x - x^*\|_2^2 + \|x - x^*\|_2^2\right) \text{ Add 0}$$

$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|x^+ - x^*\|_2^2\right)$$

$$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}\left(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$$

$f(x_k)$ is decreasing, we have $f(x_T) \leq \frac{1}{T}\sum_k f(x_k)$.

$\implies f(x_T) - f(x^*) \leq \frac{LR^2}{2T}$ where $R = \|x_0 - x^*\|$.

Also: $T = O(LR^2/\varepsilon)$ iterations for $f(x_T) - f(x^*) \leq \varepsilon$.

Gradient Descent

Primal progress

$$f(x_k) - f(x_{k+1}) \geq \frac{1}{2L} \|\nabla f(x)\|_*^2$$

Convergence

L-Lipschitz, $R = \max_{x : f(x) \leq f(x_0)} \|x - x^*\|$:

$$f(x_T) - f(x^*) \leq O(\frac{LR^2}{T})$$

To get ε-approximation, need

$$T = O(\frac{LR^2}{\varepsilon})$$

Relationship?

What is relationship to move closer to feasible?

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.
Move to other side.

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.
Move to other side.

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.
Move to other side.

What is the "hyperplane" here?

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.
Move to other side.

What is the “hyperplane” here?

$\nabla f(x)$

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.
Move to other side.

What is the "hyperplane" here?

$\nabla f(x)$ Maybe.