Strategic Games.

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Player 1: $\{$ **D**efect, **C**ooperate $\}$.
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Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

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Stable now!

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Defects! Payoff $(.1,.1)$.

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Nash Equilibrium: neither player has incentive to change strategy.

Digression..

What situations?

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Prisoner's dilemma:

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Two prisoners separated by jailors and asked to betray partner.

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More sophisticated models

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This class(today): simpler version.

Two Person Zero Sum Games

2 players.

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m strategies for player 1 n strategies for player 2

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m strategies for player 1 *n* strategies for player 2

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"Player 1 pays *a* to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

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Mixed Strategies.

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R		0	1	-1
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S		1	-1	0

How do you play?

Mixed Strategies.

		R	P	S
R	$.3\overline{3}$	0	1	-1
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How do you play?

Player 1: play each strategy with equal probability.

Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Mixed Strategies.

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Definitions.

Mixed strategies: Each player plays distribution over strategies.

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How do you play?

Player 1: play each strategy with equal probability.
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Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

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Payoffs?

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

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Average Payoff.

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Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

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Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

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$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

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Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

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$$E[X] = 0.$$

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$$E[X] = 0.[1]$$

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Will Player 1 change strategy?

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Will Player 1 change strategy? Mixed strategies uncountable!

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Expected payoffs for pure strategies for player 1.

Equilibrium

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R	$.3\overline{3}$	0	1	-1
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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Expected payoff of Scissors?

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Mixed strategy can't be better than the best pure strategy.

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Player 1 has no incentive to change! Same for player 2.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.

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Equilibrium? **(E,E)**.

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Equilibrium? **(E,E)**. Pure strategy equilibrium.

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Notation:

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Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.

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Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Playing the boss...

Row has extra strategy:Cheat.

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Ties with Rock, Paper, beats scissors.

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Payoff matrix:
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Note: column knows row cheats.

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Equilibrium: play the boss...

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Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$.

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Payoff? Remember: weighted average of pure strategies.

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Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1$

Equilibrium: play the boss...

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Payoff? Remember: weighted average of pure strategies.

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Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$

Equilibrium: play the boss...

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Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

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Both only play optimal strategies! Complementary slackness.

Why not play just one? Change payoff for other player!

Two person zero sum games.

$m \times n$ payoff matrix A.

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Payoff for strategy pair (x, y):

$$p(x, y) = x^t Ay$$

That is,

$$\sum_i x_i \left(\sum_j a_{i,j} y_j \right) = \sum_j \left(\sum_i x_i a_{i,j} \right) y_j.$$

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(No better column strategy, no better row strategy.)

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Column goes first:

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Find y, where best row is not too low..

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Example: Roshambo.

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Row goes first:
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Doesn't matter who plays first!

Equilibrium existence.

Linear programs.

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Aproximate equilibrium ...

$C(x) = \max_y x^t Ay$

$R(y) = \min_x x^t Ay$

Always: $R(y) \leq C(x)$
For $R(y)$, minimizer x "goes second", but goes first for $C(x)$.

Strategy pair: (x, y)

Equilibrium: (x, y)

$R(y) = C(x) \rightarrow C(x) - R(y) = 0$.

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Left as exercise.

Comments

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More comments

Complexity?

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Complexity?
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Versus Linear Programming: $O(n^3 m)$

More comments

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Versus Linear Programming: $O(n^3 m)$ Basically quadratic.

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Versus Linear Programming: $O(n^3 m)$ Basically quadratic.
(Faster linear programming: $O(\sqrt{n+m})$ linear system solves.)

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Still much slower ... and more complicated.

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Various assumptions: $[0,1]$ losses, other ranges takes some work.

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Runtime only dependent on m and T (number of days.)

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For each s_i, t_i, choose path p_i at random from “daily” paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

“Concentration” (law of large numbers)
$c(e)$ is relatively large $(\Omega(\log n))$
$\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results? later.

Learning

Learning just a bit.

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Example: set of labelled points, find hyperplane that separates.

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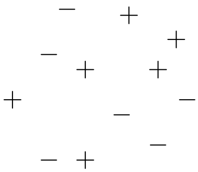

Looks hard.

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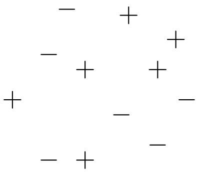

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$1/2$ of them?

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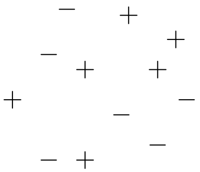

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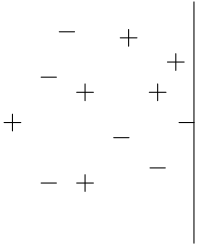

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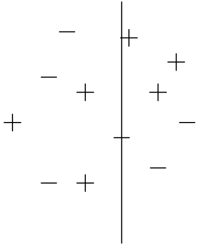

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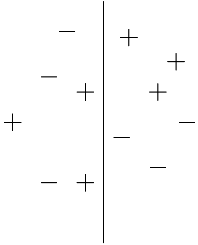

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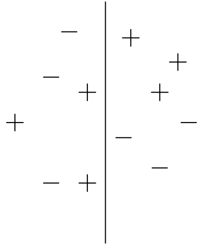

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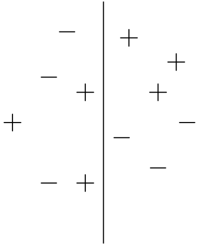

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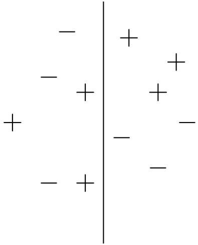

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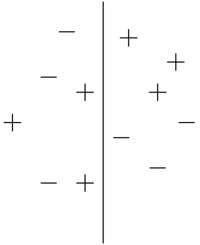

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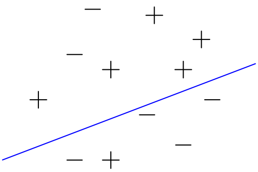

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Strong Learner:
produce hyp. correctly classifies $1 + \mu$ fraction
That's a really strong learner!

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Can one use weak learning to produce strong learner?

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Can one use weak learning to produce strong learner?

Boosting: use a weak learner to produce strong learner.

Poll.

Given a weak learning method (produce ok hypotheses.)

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Can we do this?

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(A) Yes

(B) No

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If yes.

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If yes. How?

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If yes. How?

The idea: Multiplicative Weights.

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The idea: Multiplicative Weights.
Standard online optimization method reinvented in many areas.

Boosting/MW Framework

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Really?

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Really? Proof?

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

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Second implies: $(1-\varepsilon)^x \leq e^{-\varepsilon x}$, by exponentiation.

Adaboost proof.

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$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

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Each day t, weak learner penalizes $\geq \frac{1}{2}+\gamma$ of the weight.

Loss $L_t \geq (1/2+\gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow W(T) \leq ne^{-\varepsilon \sum_t L_t} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Combining

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Every day each strategy loses (or not), $\ell_i^{(}t)$.
Assumption: Solution doesn't lose (much).

MW: keeps a distribution.

Closer?
Distance is $\sum_i \log(p_i^*/q_i)$.
Step in MW gets closer to p^* with this distance.
Idea: p^* loses less,
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Move toward playing losers less.
Thus closer to p^*.

The math:
linear (and quadratic) approximation of e^x.
Advantage?
Distributions have entropy at most $O(\log n)$.

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Online optimization: limited information.