The multiplicative weights framework.

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Sort of.

How well do you do?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

Adversary designs setup to watch who you choose, and make that expert make a mistake.

$n-1$!

Concept Alert.

Note.

Adversary:
makes you want to look bad.
"You could have done so well"...
but you didn't! ha..ha!

Analysis of Algorithms: do as well as possible!

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
Lower bound: adversary argument.
Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

What you would do anyway!

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
|"perfect" experts| drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
$\vdots$
mistake $\rightarrow$ ≤ 1 perfect expert

≥ 1 perfect expert $\rightarrow$ at most $\log n$ mistakes!

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?
each incorrect expert weight multiplied by $\frac{1}{2}$!
total weight decreases by
factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
mistake $\to \geq$ half weight with incorrect experts
($\geq \frac{1}{2}$ total.

Mistake $\to$ potential function decreased by $\frac{3}{4}$.
$\implies$ for M is number of mistakes that:

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: continued.

$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$

m - best expert mistakes M algorithm mistakes.

$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$

Take log of both sides.

$-m \leq -M \log(4/3) + \log n.$

Solve for M.

$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$

Multiple by $1-\varepsilon$ for incorrect experts...

$(1-\varepsilon)^m \leq \left(1 - \frac{\varepsilon}{2}\right)^M n.$

Massage...

$M \leq 2(1+\varepsilon)m + \frac{2 \ln n}{\varepsilon}$

Approaches a factor of two of best expert performance!

Best Analysis?

Consider two experts: A,B

Bad example?

Which is worse?

(A) A correct even days, B correct odd days

(B) A correct first half of days, B correct second

Best expert peformance: $T/2$ mistakes.

Pattern (A): $T-1$ mistakes.

Factor of (almost) two worse!

Randomization!!!!

Better approach?

Use?

Randomization!

That is, choose expert i with prob $\propto w_i$

Bad example: A,B,A,B,A...

After a bit, A and B make nearly the same number of mistakes.

Choose each with approximately the same probabilty.

Make a mistake around $1/2$ of the time.

Best expert makes $T/2$ mistakes.

Roughly optimal!

Randomized analysis.

Some formulas:

For $\varepsilon \leq \frac{1}{2}, x \in [0,1]$,

$(1-\varepsilon)^x \leq (1 - \varepsilon x)$

For $\varepsilon \in [0, \frac{1}{2}]$,

$-\varepsilon - \varepsilon^2 \leq \ln(1-\varepsilon) \leq -\varepsilon$

$\varepsilon - \varepsilon^2 \leq \ln(1+\varepsilon) \leq \varepsilon$

Proof Idea: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Randomized algorithm

Expert i loses $\ell_i^t \in [0,1]$ in round t.

1. Initially $w_i = 1$ for expert i.
2. Choose expert i with prob $\frac{w_i}{W}$, $W = \sum_i w_i$.
3. $w_i \leftarrow w_i(1-\varepsilon)^{\ell_i^t}$

$W(t)$ sum of w_i at time t. $W(0) = n$

Best expert, b, loses L^* total. $\rightarrow W(T) \geq w_b \geq (1-\varepsilon)^{L^*}$.

$L_t = \sum_i \frac{w_i \ell_i^t}{W}$ expected loss of alg. in time t.

Claim: $W(t+1) \leq W(t)(1-\varepsilon L_t)$ Loss $\rightarrow$ weight loss.

Proof:

$$W(t+1) = \sum_i (1-\varepsilon)^{\ell_i^t} w_i \leq \sum_i (1-\varepsilon\ell_i^t) w_i = \sum_i w_i - \varepsilon \sum_i w_i \ell_i^t = \sum_i w_i \left(1 - \varepsilon \frac{\sum_i w_i \ell_i^t}{\sum_i w_i}\right) = W(t)(1-\varepsilon L_t)$$

Analysis

$$(1-\varepsilon)^{L^*} \leq W(T) \leq n \ \prod_t (1-\varepsilon L_t)$$

Take logs

$$(L^*)\ln(1-\varepsilon) \leq \ln n + \sum \ln(1-\varepsilon L_t)$$

Use $-\varepsilon - \varepsilon^2 \leq \ln(1-\varepsilon) \leq -\varepsilon$

$$-(L^*)(\varepsilon + \varepsilon^2) \leq \ln n - \varepsilon \sum L_t$$

And

$$\sum_t L_t \leq (1+\varepsilon)L^* + \frac{\ln n}{\varepsilon}.$$

$\sum_t L_t$ is total expected loss of algorithm.

Within $(1+\varepsilon)$ ish of the best expert!

No factor of 2 loss!

Gains.

Why so negative?

Each day, each expert gives gain in $[0,1]$.

Multiplicative weights with $(1+\varepsilon)^{g_i^t}$.

$$G \geq (1-\varepsilon)G^* - \frac{\log n}{\varepsilon}$$

where G^* is payoff of best expert.

Scaling:

Not $[0,1]$, say $[0,\rho]$.

$$L \leq (1+\varepsilon)L^* + \frac{\rho \log n}{\varepsilon}$$

Summary: multiplicative weights.

Framework: n experts, each loses different amount every day.

Perfect Expert: $\log n$ mistakes.

Imperfect Expert: best makes m mistakes.

Deterministic Strategy: $2(1+\varepsilon)m + \frac{\log n}{\varepsilon}$

Real numbered losses: Best loses L^* total.

Randomized Strategy: $(1+\varepsilon)L^* + \frac{\log n}{\varepsilon}$

Strategy:

Choose proportional to weights

multiply weight by $(1-\varepsilon)^{\text{loss}}$.

Multiplicative weights framework!

Applications next!

Strategic Games.

N players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(.1.1)

What is the best thing for the players to do?

Both cooperate. Payoff $(3,3)$.

If player 1 wants to do better, what do they do?

Defects! Payoff $(5,0)$

What does player 2 do now?

Defects! Payoff $(.1,.1)$.

Stable now!

Nash Equilibrium: neither player has incentive to change strategy.

Digression..

What situations?

Prisoner's dilemma:
Two prisoners separated by jailors and asked to betray partner.

Basis of the free market.
Companies compete, don't cooperate.
No Monopoly:
E.G., OPEC, Airlines, .
Should defect.
Why don't they?
Free market economics ...not so much?
More sophisticated models ,e.g, iterated dominance, coalitions, complexity..
Lots of interesting Game Theory!

This class(today): simpler version.

Two Person Zero Sum Games

2 players.

Each player has strategy set:
m strategies for player 1 n strategies for player 2

Payoff function: $u(i,j) = (-a, a)$ (or just a).
"Player 1 pays a to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by m by n matrix: A.
Row player minimizes, column player maximizes.

Roshambo: rock,paper, scissors.

	R	P	S
R	0	1	-1
P	-1	0	1
S	1	-1	0

Any Nash Equilibrium?

(R,R)? no. (R,P)? no. (R,S)? no.

Mixed Strategies.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = 0.^{1}$$

[1] Remember zero sum games have one payoff.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: column knows row cheats.
Why play?
Row is column's advisor.
... boss.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$
Strategy 3: $\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}$
Strategy 4: $\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = -\frac{1}{6}$

Payoff is $0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6}$

Column player: every column payoff is $-\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.

Why not play just one? Change payoff for other player!

Two person zero sum games.

$m \times n$ payoff matrix A.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair (x, y):

$$p(x,y) = x^t A y$$

That is,

$$\sum_i x_i \left(\sum_j a_{i,j} y_j \right) = \sum_j \left(\sum_i x_i a_{i,j} \right) y_j.$$

Recall row minimizes, column maximizes.

Equilibrium pair: (x^*, y^*)?

$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

Equilibrium.

Equilibrium pair: (x^*, y^*)?

$$p(x,y) = (x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

No row is better:
$\min_i A^{(i)} \cdot y = (x^*)^t A y^*$. [2]

No column is better:
$\max_j (A^t)^{(j)} \cdot x = (x^*)^t A y^*$.

[2] $A^{(i)}$ is ith row.

Best Response

Column goes first:
Find y, where best row is not too low..

$$R = \max_y \min_x (x^t A y).$$

Note: x can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Agin: y of form $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of C?

Duality.

$$R = \max_y \min_x (x^t A y).$$
$$C = \min_x \max_y (x^t A y).$$

Weak Duality: $R \leq C$.
Proof: Better to go second.
Blindly play go-first strategy. □

At Equilibrium (x^*, y^*), payoff v:
row payoffs (Ay^*) all $\geq v \implies R \geq v$.
column payoffs $((x^*)^t A)$ all $\leq v \implies v \geq C$.
$\implies R \geq C$

Equilibrium $\implies R = C$!

Strong Duality: There is an equilibrium point! and $R = C$!

Doesn't matter who plays first!

Equilibrium existence.

Linear programs.

Column player: find y to maximize row payoffs.
$\max z, Ay \geq z, \sum_i y_i = 1$
Row player: find x to minimize column payoffs.
$\min z, A^T x \leq z, \sum_i x_i = 1$.

Primal dual optimal are equilibrium solution.

Strong Duality: linear program.

Aproximate equilibrium ...

$C(x) = \max_y x^t Ay$

$R(y) = \min_x x^t Ay$

Always: $R(y) \leq C(x)$
For $R(y)$, minimizer x "goes second", but goes first for $C(x)$.

Strategy pair: (x, y)

Equilibrium: (x, y)

$R(y) = C(x) \to C(x) - R(y) = 0$.

Approximate Equilibrium: $C(x) - R(y) \leq \varepsilon$.

With $R(y) < C(x)$
$\to$ "Response y to x is within ε of best response"
$\to$ "Response x to y is within ε of best response"

Games and experts

Again: find (x^*, y^*), such that
$(\max_y x^* Ay) - (\min_x xAy^*) \leq \varepsilon$
$C(x^*) \quad - \quad R(y^*) \leq \varepsilon$

Experts Framework:
n Experts, T days, L^* -total loss of best expert.

Multiplicative Weights Method yields loss L where
$L \leq (1+\varepsilon)L^* + \frac{\log n}{\varepsilon}$

Games and Experts.

Assume: A has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) m pure row strategies are experts.
Use multiplicative weights, produce row distribution.

Let x_t be distribution (row strategy) on day t.

2) Each day, adversary plays best column response to x_t.
Choose column of A that maximizes row's expected loss.

Let y_t be indicator vector for "best" response column.

Approximate Equilibrium!

Experts: x_t is strategy on day t, y_t is best column against x_t.

Let $y^* = \frac{1}{T}\sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t A y_t$.

Claim: (x^*, y^*) are 2ε-optimal for matrix A.

Column payoff: $C(x^*) = \max_y x^* Ay$.
Loss on day t, $x_t A y_t \geq x^* A y_{t^*} = C(x^*)$ by the choice of x^* .
Thus, algorithm loss, L, is $\geq T \times C(x^*)$.

Best expert: L^*- best row against all the columns played.

best row against $\sum_t Ay_t$ and $T \times y^* = \sum_t y_t$
$\to$ best row against $T \times Ay^*$.
$\to L^* \leq T \times R(y^*)$.

Multiplicative Weights: $L \leq (1+\varepsilon)L^* + \frac{\ln n}{\varepsilon}$

$T \times C(x^*) \leq (1+\varepsilon) T \times R(y^*) + \frac{\ln n}{\varepsilon} \to C(x^*) \leq (1+\varepsilon) R(y^*) + \frac{\ln n}{\varepsilon T}$
$\to C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T}$.

$T = \frac{\ln n}{\varepsilon^2}$, $R(y^*) \leq 1$
$\to C(x^*) - R(y^*) \leq 2\varepsilon$.

Approximate Equilibrium: slightly different!

Experts: x_t is strategy on day t, y_t is best column against x_t.

Let $x^* = \frac{1}{T}\sum_t x_t$ and $y^* = \frac{1}{T}\sum_t y_t$.

Claim: (x^*, y^*) are 2ε-optimal for matrix A.

Left as exercise.

Comments

For any ε, there exists an ε-Approximate Equilibrium.
Does an equilibrium exist? Yes.

Something about math here?
Limit of a sequence on some closed set..hmmm..

More comments

Complexity?
$T = \frac{\ln n}{\varepsilon^2} \rightarrow O(nm\frac{\log n}{\varepsilon^2})$. Basically linear!

Versus Linear Programming: $O(n^3 m)$ Basically quadratic.
(Faster linear programming: $O(\sqrt{n+m})$ linear system solves.)
Still much slower ... and more complicated.

Dynamics: best response, update weight according to loss, ...

Near integrality.
Only $\ln n/\varepsilon^2$ non-zero column variables.
Average $1/T$, so not too many nonzeros and not too small.

Not stochastic at all here, the column responses are adversarial.

Various assumptions: $[0,1]$ losses, other ranges takes some work.

Toll/Congestion

Given: $G = (V,E)$.
Given $(s_1,t_1)\ldots(s_k,t_k)$.

Problem: Route path for each pair and minimize maximum congestion.

Congestion is maximum number of paths that use any edge.

Note: Number of paths is exponential.
Can encode in polysized linear program, but large.

Toll/Congestion

Given: $G = (V,E)$.
Given $(s_1,t_1)\ldots(s_k,t_k)$.
Row: choose routing of all paths. (Exponential)
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r,e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max congestion on any edge.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e,r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Multiplicative Weights only maintains m weights.

Adversary only needs to provide best column each day.

Runtime only dependent on m and T (number of days.)

Congestion minimization and Experts.

Will use gain and $[0,\rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon/(1-\varepsilon)$.

Proof:
$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

$G^* = T * c_{\max}$ – Best row payoff against average routing (times T).

$G \leq T \times C^*$ – each day, gain is avg. congestion $\leq$ opt congestion.

$T = \frac{k \log n}{\varepsilon^2} \rightarrow Tc_{\max} - TC^* \leq \varepsilon Tc_{\max} + \frac{k \log n}{\varepsilon} \quad \rightarrow$
$c_{max} - C^* \leq \varepsilon c_{max} + \varepsilon$ □

Better setup.

Runtime: $O(km\log n)$ to route in each step (using Dijkstra's)

$O(\frac{k\log n}{\varepsilon^2})$ steps
to get $c_{max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.

To get constant c error.
$\rightarrow O(k^2 m\log n/\varepsilon^2)$ to get a constant approximation.
Exercise: $O(km\log n/\varepsilon^2)$ algorithm ! ! !

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

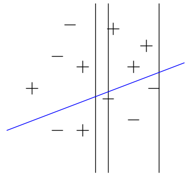
"Concentration" (law of large numbers)
$c(e)$ is relatively large ($\Omega(\log n)$)
$\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results? later.

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

$1/2$ of them? Easy.
Arbitrary line. And Scan.

Useless. A bit more than $1/2$ Correct would be better.

Weak Learner: Classify $\geq \frac{1}{2}+\varepsilon$ points correctly.

Not really important but ...

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2}+\varepsilon$ fraction

Strong Learner:
produce hyp. correctly classifies $1+\mu$ fraction

That's a really strong learner!
Strong Learner:
produce hypothesis correctly classifies $1-\mu$ fraction

Same thing?

Can one use weak learning to produce strong learner?

Boosting: use a weak learner to produce strong learner.

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes

(B) No

If yes. How?

The idea: Multiplicative Weights.
Standard online optimization method reinvented in many areas.

Boosting/MW Framework

Points lose when classified correctly.
The little devils want to fool the learner.

Learner classifies weighted majority of points correctly.

Strong learner algorithm from many weak learners!

Initialize: all points have weight 1.

Do $T = \frac{2}{\varepsilon^2}\ln\frac{1}{\mu}$ rounds

1. Find $h_t(\cdot)$ correct on $1/2+\gamma$ of weighted points.
2. Multiply each point that is correct by $(1-\varepsilon)$.

Output hypotheses $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1-\mu$ of the points ! ! !

Cool!

Really? Proof?

Logarithm

$ln(1-x) = (-x - x^2/2 - x^3/3....)$ Taylors formula for $|x| < 1$.

Implies: for $x \leq 1/2$, that $-x - x^2 \leq \ln(1-x) \leq -x$.

The first inequality is from geometric series.

$x^3/3 + ... = x^2(x/3 + x^2/4 + ..) \leq x^2(1/2)$ for $|x| < 1/2$.

The second is from truncation.

Second implies: $(1-\varepsilon)^x \leq e^{-\varepsilon x}$, by exponentiation.

Some details...

Weak learner learns over distributions of points not points.

Make copies of points to simulate distributions.

Used often in machine learning.
Blending learning methods.

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points!

Let S_{bad} be the set of points where $h(x)$ is incorrect.

majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

point $x \in S_{bad}$ is winning – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$

Each day t, weak learner penalizes $\geq \frac{1}{2} + \gamma$ of the weight.

Loss $L_t \geq (1/2 + \gamma)$

$\rightarrow W(t+1) \leq W(t)(1-\varepsilon(L_t)) \leq W(t)e^{-\varepsilon L_t}$

$\rightarrow W(T) \leq ne^{-\varepsilon \sum_t L_t} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Combining

$|S_{bad}|(1-\varepsilon)^{T/2} \leq W(T) \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon = \gamma$, take logs.

$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$

Again, $-\gamma - \gamma^2 \leq \ln(1-\gamma)$,

$\ln\left(\frac{|S_{bad}|}{n}\right) + \frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$

And $T = \frac{2}{\gamma^2}\log\mu$,

$\rightarrow \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \rightarrow \frac{|S_{bad}|}{n} \leq \mu$.

The misclassified set is at most μ fraction of all the points.

The hypothesis correctly classifies $1-\mu$ of the points !

Claim: Multiplicative weights: $h(x)$ is correct on $1-\mu$ of the points!