Today

Other algorithms.

Today

Other algorithms.

For linear programming.

Today

Other algorithms.

For linear programming.

Online.

Perceptron Guarantees.

Separable set of points.

Perceptron Guarantees.

Separable set of points.

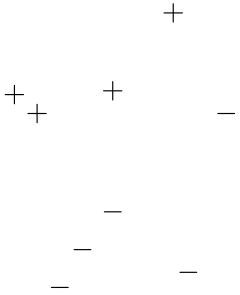
Perceptron.

Perceptron Guarantees.

Separable set of points.

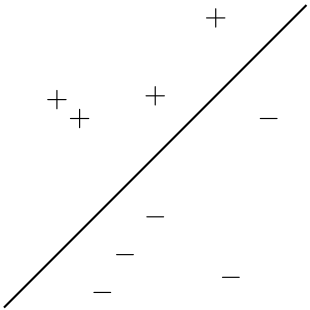
Perceptron.
Prove a performance bound.

Margin and Perceptron



Labelled points with $x_1, \ldots, x_n$.

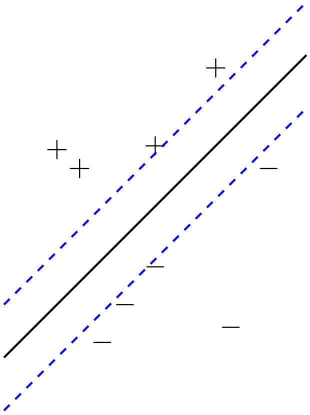
Margin and Perceptron



Labelled points with $x_1, \ldots, x_n$.

Hyperplane separator.

Margin and Perceptron



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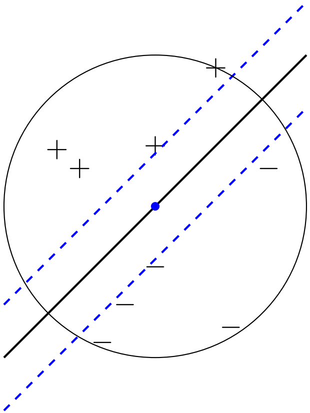
Hyperplane separator.

Margins.

Margin and Perceptron



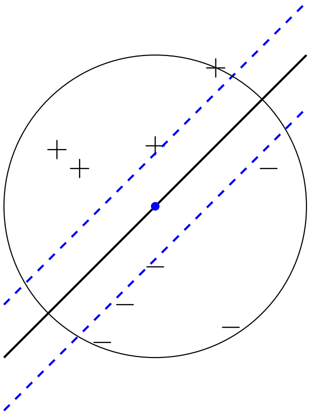
Labelled points with $x_1, \ldots, x_n$.

Hyperplane separator.

Margins.

Inside unit ball.

Margin and Perceptron



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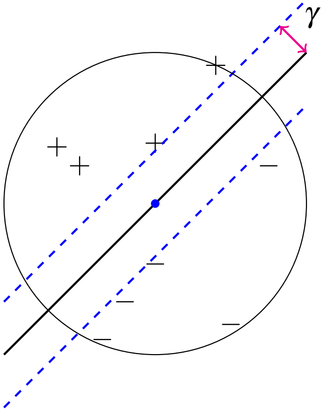

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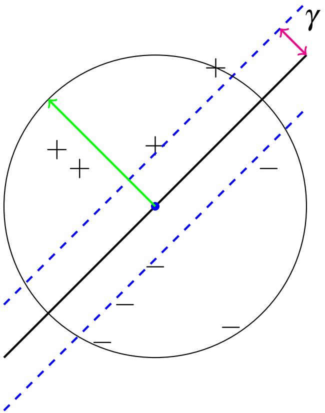
Margins.

Inside unit ball.
Margin γ

Margin and Perceptron



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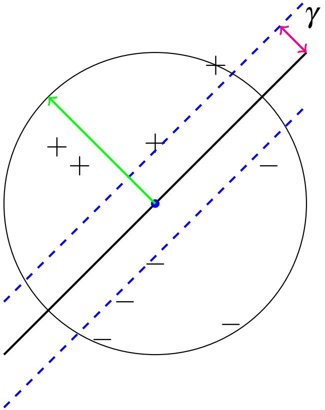
Margin and Perceptron



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Margins.

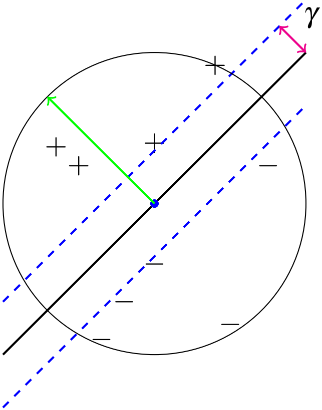
Inside unit ball.

Margin γ

Hyperplane:

$w \cdot x \geq \gamma$ for $+$ points.

Margin and Perceptron



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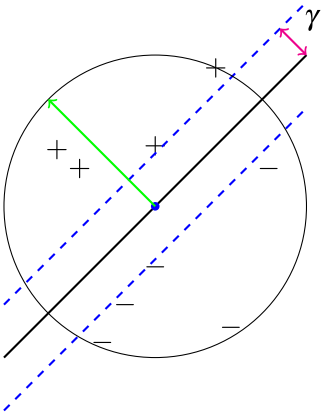

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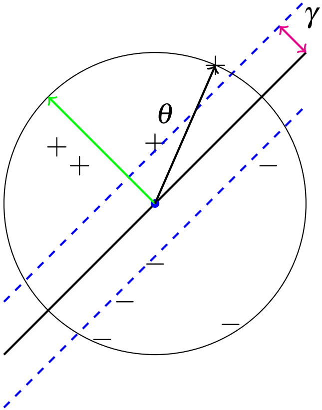
Put points on unit ball.

Margin and Perceptron



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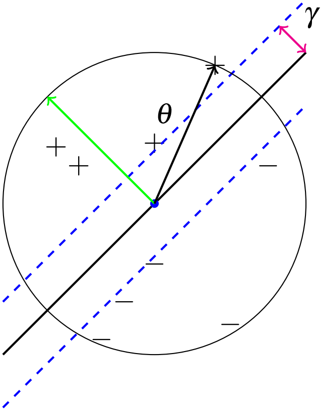
Put points on unit ball.

$w \cdot x = cos\theta$

Margin and Perceptron



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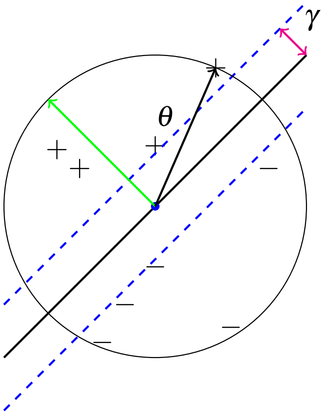
Will assume positive labels!

Margin and Perceptron



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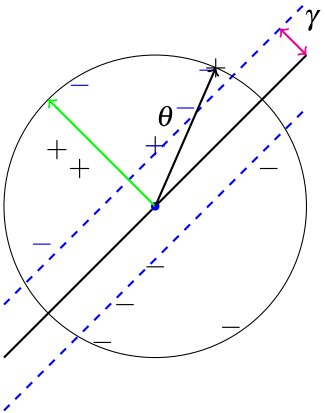
negate the negative:

$(x, -1) \rightarrow (-x, 1)$

Margin and Perceptron



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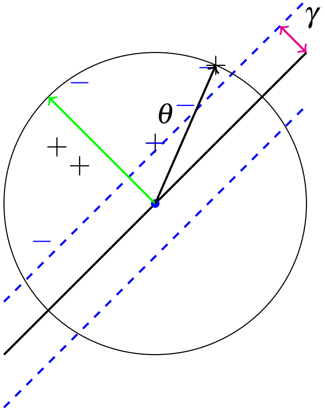

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Perceptron Algorithm

An aside: a hyperplane is a perceptron.

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Alg: Given $x_1, \ldots, x_n$**.**

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Theorem: Algorithm only makes $\frac{1}{\gamma^2}$ mistakes.

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A step in the right direction!

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Claim 1: $w_{t+1} \cdot w \geq w_t \cdot w + \gamma$.

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A γ *in the right direction!*

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A γ *in the right direction!* w_{t+1} is more like w.

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□

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Claim 2: $|w_{t+1}|^2 \leq |w_t|^2 + 1$

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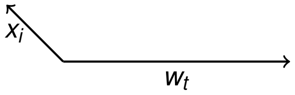

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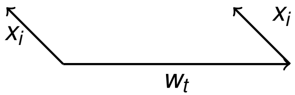

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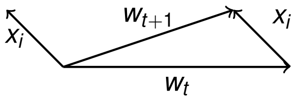
$$w_{t+1} = w_t + x_i$$



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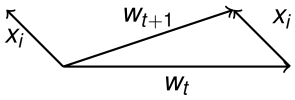
Claim 2: $|w_{t+1}|^2 \leq |w_t|^2 + 1$

$w_{t+1} = w_t + x_i$
Less than a right angle!



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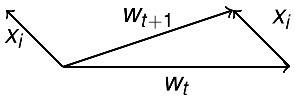

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Less than a right angle!
$\rightarrow |w_{t+1}|^2 \leq |w_t|^2 + |x_i|^2 \leq |w_t|^2 + 1.$

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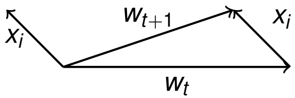

$w_{t+1} = w_t + x_i$
Less than a right angle!
$\rightarrow |w_{t+1}|^2 \leq |w_t|^2 + |x_i|^2 \leq |w_t|^2 + 1.$
Algebraically.

Proof:continued.

Alg: Given $x_1, \ldots, x_n$**.**

Let $w_1 = x_1$.
For each x_i where $w_t \cdot x_i$ has wrong sign (negative) Mistake
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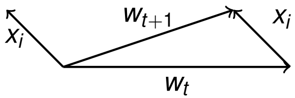

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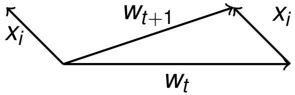

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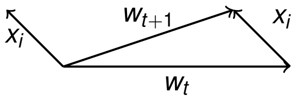

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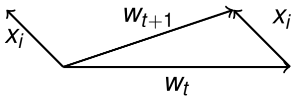

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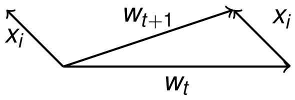

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□

Claim 2 holds even if no separating hyperplane!

Putting it together...

Claim 1: $w_{t+1} \cdot w \geq w_t \cdot w + \gamma$.

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Let $t = M$.

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γM

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$$\gamma M \leq w_M \cdot w$$
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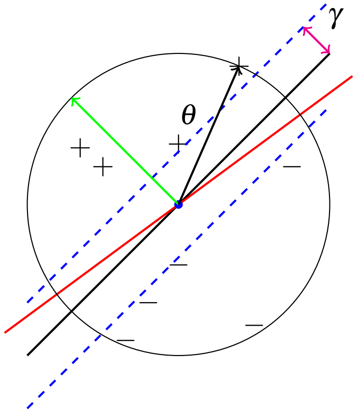
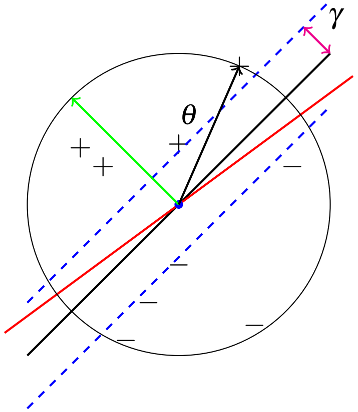
$\rightarrow M \leq \frac{1}{\gamma^2}$

Finding fat separator.



There is a γ separating hyperplane.

Finding fat separator.



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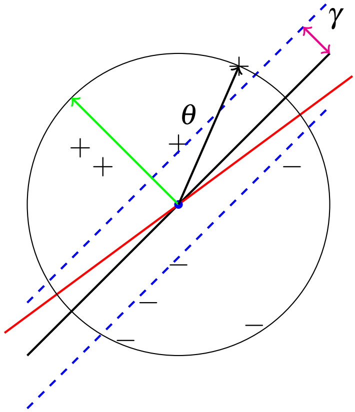

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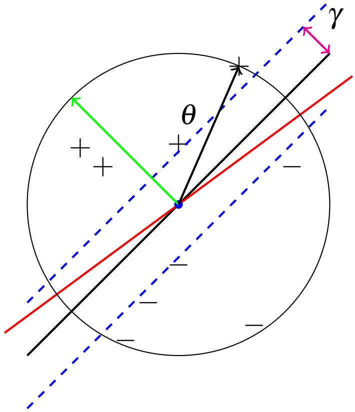

There is a γ separating hyperplane.

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Does perceptron find big margin separator.

Approximately Maximizing Margin Algorithm

There is a γ separating hyperplane.

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Any point within $\gamma/2$ is still a mistake.

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Same (ish) as before.

Margin Approximation: Claim 2

Claim 2(?): $|w_{t+1}|^2 \leq |w_t|^2 + 1$??

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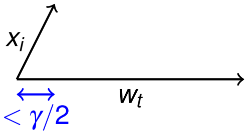
Adding x_i to w_t even if in correct direction.



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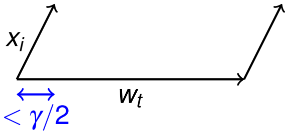

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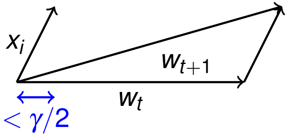

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Obtuse triangle.

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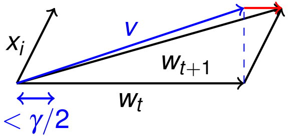

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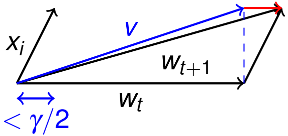

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$|v|^2 \leq |w_t|^2 + 1$

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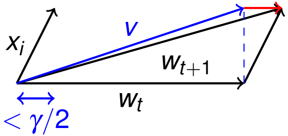

Adding x_i to w_t even if in correct direction.

Obtuse triangle.

$|v|^2 \leq |w_t|^2 + 1$
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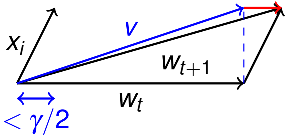

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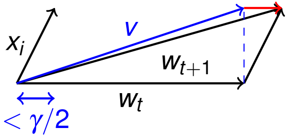

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Red bit is at most $\gamma/2$.

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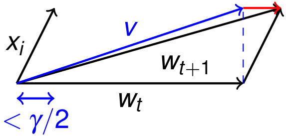

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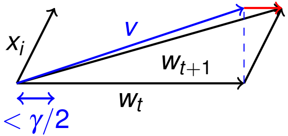

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If $|w_t| \geq \frac{2}{\gamma}$, then $|w_{t+1}| \leq |w_t| + \frac{3}{4}\gamma$.

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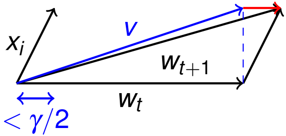

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M updates

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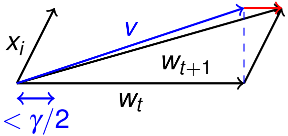

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M updates $|w_M| \leq \frac{2}{\gamma} + \frac{3}{4}\gamma M$.

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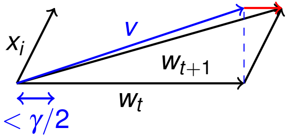

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Claim 1:

Margin Approximation: Claim 2

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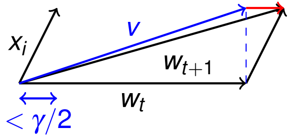

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M updates $|w_M| \leq \frac{2}{\gamma} + \frac{3}{4}\gamma M$.

Claim 1: Implies $|w_M| \geq \gamma M$.

Margin Approximation: Claim 2

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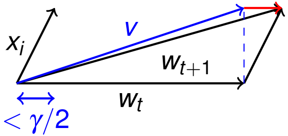

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Together: $|w_{t+1}| \leq |w_t| + \frac{1}{2|w_t|} + \frac{\gamma}{2}$

If $|w_t| \geq \frac{2}{\gamma}$, then $|w_{t+1}| \leq |w_t| + \frac{3}{4}\gamma$.

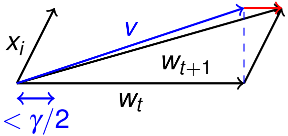
M updates $|w_M| \leq \frac{2}{\gamma} + \frac{3}{4}\gamma M$.

Claim 1: Implies $|w_M| \geq \gamma M$.

$\gamma M \leq \frac{2}{\gamma} + \frac{3}{4}\gamma M$

Margin Approximation: Claim 2

Claim 2(?): $|w_{t+1}|^2 \leq |w_t|^2 + 1$??



Adding x_i to w_t even if in correct direction.

Obtuse triangle.

$|v|^2 \leq |w_t|^2 + 1$
$\rightarrow |v| \leq |w_t| + \frac{1}{2|w_t|}$
(square right hand side.)

Red bit is at most $\gamma/2$.

Together: $|w_{t+1}| \leq |w_t| + \frac{1}{2|w_t|} + \frac{\gamma}{2}$

If $|w_t| \geq \frac{2}{\gamma}$, then $|w_{t+1}| \leq |w_t| + \frac{3}{4}\gamma$.

M updates $|w_M| \leq \frac{2}{\gamma} + \frac{3}{4}\gamma M$.

Claim 1: Implies $|w_M| \geq \gamma M$.

$\gamma M \leq \frac{2}{\gamma} + \frac{3}{4}\gamma M \rightarrow M \leq \frac{8}{\gamma^2}$

The multiplicative weights framework.

Experts framework.

n experts.

Experts framework.

n experts.

Every day, each offers a prediction.

Experts framework.

n experts.

Every day, each offers a prediction.

“Rain” or “Shine.”

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1				$\cdots$	
Expert 2				$\cdots$	
Expert 3				$\cdots$	
$\vdots$				$\cdots$	

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine			$\cdots$	
Expert 2	Shine			$\cdots$	
Expert 3	Rain			$\cdots$	
$\vdots$	$\vdots$			$\cdots$	

Rained!

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain		$\cdots$	
Expert 2	Shine	Shine		$\cdots$	
Expert 3	Rain	Rain		$\cdots$	
$\vdots$	$\vdots$	$\vdots$		$\cdots$	

Rained! Shined!

Experts framework.

n experts.

Every day, each offers a prediction.

“Rain” or “Shine.”

	Day 1	Day 2	Day 3	⋯	Day T
Expert 1	Shine	Rain	Shine	⋯	
Expert 2	Shine	Shine	Shine	⋯	
Expert 3	Rain	Rain	Rain	⋯	
⋮	⋮	⋮	Shine	⋯	

Rained! Shined! Shined!

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	···	Day T
Expert 1	Shine	Rain	Shine	···	
Expert 2	Shine	Shine	Shine	···	
Expert 3	Rain	Rain	Rain	···	
⋮	⋮	⋮	Shine	···	

Rained! Shined! Shined! ···

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	⋯	Day T
Expert 1	Shine	Rain	Shine	⋯	
Expert 2	Shine	Shine	Shine	⋯	
Expert 3	Rain	Rain	Rain	⋯	
⋮	⋮	⋮	Shine	⋯	

Rained! Shined! Shined! ⋯

Whose advice do you follow?

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Sort of.

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Sort of.

How well do you do?

Infallible expert.

One of the experts is infallible!

Infallible expert.

One of the experts is infallible!

Your strategy?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never!

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1
(B) 2
(C) $\log n$
(D) $n-1$

Adversary designs setup to watch who you choose, and make that expert make a mistake.

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1
(B) 2
(C) $\log n$
(D) $n-1$

Adversary designs setup to watch who you choose, and make that expert make a mistake.

$n-1$!

Concept Alert.

Note.

Concept Alert.

Note.

Adversary:

Concept Alert.

Note.

Adversary:
makes you want to look bad.

Concept Alert.

Note.

Adversary:
makes you want to look bad.
”You could have done so well”...

Concept Alert.

Note.

Adversary:
makes you want to look bad.
”You could have done so well”...
but you didn’t!

Concept Alert.

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Adversary:
makes you want to look bad.
”You could have done so well”...
but you didn’t! ha..

Concept Alert.

Note.

Adversary:

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but you didn’t! ha..ha!

Concept Alert.

Note.

Adversary:
makes you want to look bad.
”You could have done so well”...
but you didn’t! ha..ha!

Analysis of Algorithms: do as well as possible!

Back to mistake bound.

Infallible Experts.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$

Lower bound: adversary argument.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound:

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
Lower bound: adversary argument.
Upper bound: every mistake finds fallible expert.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
Lower bound: adversary argument.
Upper bound: every mistake finds fallible expert.

Better Algorithm?

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

What you would do anyway!

Alg 2: find majority of the perfect

How many mistakes could you make?

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n - 1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow \quad \leq n/2$ perfect experts
mistake $\rightarrow \quad \leq n/4$ perfect experts

$\vdots$

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
$\vdots$
mistake $\rightarrow$ ≤ 1 perfect expert

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n - 1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
⋮
mistake $\rightarrow$ ≤ 1 perfect expert

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
$\vdots$
mistake $\rightarrow$ ≤ 1 perfect expert

≥ 1 perfect expert

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
$\vdots$
mistake $\rightarrow$ ≤ 1 perfect expert

≥ 1 perfect expert $\rightarrow$ at most $\log n$ mistakes!

Imperfect Experts

Goal?

Imperfect Experts

Goal?

Do as well as the best expert!

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Analysis: weighted majority

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function:

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
 total weight of incorrect experts reduced by

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

total weight of incorrect experts reduced by

-1? -2? factor of $\frac{1}{2}$?

each incorrect expert weight multiplied by $\frac{1}{2}$!

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

total weight of incorrect experts reduced by

-1? -2? factor of $\frac{1}{2}$?

each incorrect expert weight multiplied by $\frac{1}{2}$!

total weight decreases by

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

total weight of incorrect experts reduced by

-1? -2? factor of $\frac{1}{2}$?

each incorrect expert weight multiplied by $\frac{1}{2}$!

total weight decreases by

factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

total weight of incorrect experts reduced by

-1? -2? factor of $\frac{1}{2}$?

each incorrect expert weight multiplied by $\frac{1}{2}$!

total weight decreases by

factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?

mistake $\rightarrow \geq$ half weight with incorrect experts

($\geq \frac{1}{2}$ total.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?
each incorrect expert weight multiplied by $\frac{1}{2}$!
total weight decreases by
factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
mistake $\rightarrow \geq$ half weight with incorrect experts
($\geq \frac{1}{2}$ total.

Mistake $\rightarrow$ potential function decreased by $\frac{3}{4}$.
$\Longrightarrow$ for M is number of mistakes that:

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Solve for M.

$M \leq (m + \log n)/\log(4/3)$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$-m \leq -M \log(4/3) + \log n.$

Solve for M.

$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$

Analysis: continued.

$$\frac{1}{2^m} \le \sum_i w_i \le \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \le \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \le -M \log(4/3) + \log n.$$

Solve for M.

$$M \le (m + \log n)/\log(4/3) \le 2.4(m + \log n)$$

Multiple by $1 - \varepsilon$ for incorrect experts...

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Solve for M.

$$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$$

Multiple by $1-\varepsilon$ for incorrect experts...

$$(1-\varepsilon)^m \leq \left(1 - \frac{\varepsilon}{2}\right)^M n.$$

Analysis: continued.

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Approaches a factor of two of best expert performance!

Best Analysis?

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Consider two experts: A,B

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Which is worse?

(A) A correct even days, B correct odd days
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Factor of (almost) two worse!

Randomization

Better approach?

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Use?

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Choose each with approximately the same probabilty.

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Proof Idea: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

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Analysis

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No factor of 2 loss!

Gains.

Why so negative?

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Multiplicative weights with $(1+\varepsilon)^{g_i^t}$.

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Summary: multiplicative weights.

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    multiply weight by $(1-\varepsilon)^{\text{loss}}$.

Summary: multiplicative weights.

Framework: n experts, each loses different amount every day.

Perfect Expert: $\log n$ mistakes.

Imperfect Expert: best makes m mistakes.

Deterministic Strategy: $2(1+\varepsilon)m + \frac{\log n}{\varepsilon}$

Real numbered losses: Best loses L^* total.

Randomized Strategy: $(1+\varepsilon)L^* + \frac{\log n}{\varepsilon}$

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Choose proportional to weights
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Multiplicative weights framework!

Applications next!

Strategic Games.

N players.

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Each player has strategy set. $\{S_1, \ldots, S_N\}$.

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Player 1: $\{$ **D**efect, **C**ooperate $\}$.
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Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

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Nash Equilibrium: neither player has incentive to change strategy.

Digression..

What situations?

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Prisoner's dilemma:

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Two prisoners separated by jailors and asked to betray partner.

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This class(today): simpler version.

Two Person Zero Sum Games

2 players.

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m strategies for player 1 n strategies for player 2

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"Player 1 pays *a* to player 2."

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Mixed Strategies.

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S		1	-1	0

How do you play?

Mixed Strategies.

		R	P	S
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

How do you play?

Player 1: play each strategy with equal probability.

Mixed Strategies.

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Player 2: play each strategy with equal probability.

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Mixed strategies: Each player plays distribution over strategies.

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Player 1: play each strategy with equal probability.
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Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

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[1] Remember zero sum games have one payoff.

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Average Payoff.

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Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
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Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

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Will Player 1 change strategy?

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Will Player 1 change strategy? Mixed strategies uncountable!

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Expected payoffs for pure strategies for player 1.

Equilibrium

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Expected payoff of Rock?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j)$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
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No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)}(Pr[i] \times Pr[j])X(i,j) = \sum_i Pr[i](\sum_j Pr[j] \times X(i,j))$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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S	$.3\overline{3}$	1	-1	0

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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)}(Pr[i] \times Pr[j])X(i,j) = \sum_i Pr[i](\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

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Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
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Mixed strategy can't be better than the best pure strategy.

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Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.

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PreEmpt ties preEmpt, beats everything else.

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Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium?

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**.

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Notation:

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Playing the boss...

Row has extra strategy:Cheat.

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: column knows row cheats.

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

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Note: column knows row cheats.
Why play?

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Why play?
Row is column's advisor.

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

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Note: column knows row cheats.
Why play?
Row is column's advisor.
... boss.

Playing the boss...

Row has extra strategy:Cheat.
Ties with Rock, Paper, beats scissors.
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: column knows row cheats.
Why play?
Row is column's advisor.
... boss.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff?

Equilibrium: play the boss...

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Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

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Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$
Strategy 3: $\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

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$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

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Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$
Strategy 3: $\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}$
Strategy 4: $\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
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Equilibrium: play the boss...

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Column player: every column payoff is $-\frac{1}{6}$.

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Both only play optimal strategies! Complementary slackness.

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Why not play just one?

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Column player: every column payoff is $-\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.

Why not play just one? Change payoff for other guy!

Next time: Multiplicative weights and games.