Interior point on the central path.

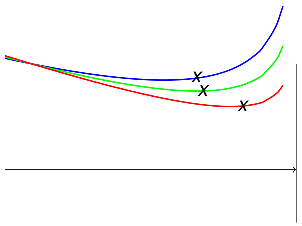
Find x, that minimizes $f_0(x)$ subject to

$f_i(x) \leq 0, i = 1, \ldots m.$

Central path:

$\min tf_0(x) - \sum_{i=1}^{m} \ln(-f_i(x))$

The minimizer, $x(t)$, form the **central path.**



The sequence of x's are "central path".

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition? Take Derivative.

$t\nabla f_0(x) - \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$ $\nabla f_0(x) - \sum_{i=1} \frac{1}{tf_i(x)} \nabla f_i(x) = 0$

Or, $\nabla f_0(x) = \sum_{i=1} \frac{\nabla f_i(x)}{tf_i(x)}$ (Opposing force fields.)

Recall, Lagrangian: $L(\lambda, x) = f_0(x) + \sum_i \lambda_i f_i(x)$.

Fix λ, optimize for x^* give valid lower bound on solution.

Optimality Condition.

Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0.$

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Value? Found λ where:

$\min_x L(\lambda, x) = f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$

Central point $x(t)$ within $\frac{m}{t}$ of optimal primal!!!!

$L(\lambda, x(t)) \geq f_0(x) - \frac{m}{t} \implies \min_x L(\lambda, x) + \frac{m}{t} \geq f_0(x)$

$\implies OPT + \frac{m}{t} \geq f_0(x)$

Central path.

$\min_x f_0(x), f_i(x) \leq 0.$

$\min_x tf_0(x) - \sum_{i>0} \ln(-f_i(x))$

Optimal: $x(t)$ is feasible.

$f_0(x(t)) \leq OPT + \frac{m}{t}$

Algorithm: take $t \to \infty$.

Finding $x(t)$?

Assume you have $x(t)$, change $t = \mu t$, for $\mu > 1$.

Find $x(\mu t)$.

Idea: newton's method.

Should show new optimal point not too different from old.

Next.

Central Path evolution.

Old point $x = x(t)$ versus $x^+ = x(\mu t)$?

Minimizing: $\mu tf_0(x) - \sum_{i=1}^{n} \ln(-f_i(t))$.

Difference in new objective from old optimal point to new:

$\mu tf_0(x) - \sum_{i=1}^{m} \ln(-f_i(x)) - \mu tf_0(x^+) + \sum_{i=1}^{m} \ln(-f_i(x^+))$

Simplify: $\mu tf_0(x) - \mu tf_0(x^+) + \sum_i \ln(\frac{f_i(x^+)}{f_i(x)})$

Let $\lambda_i = -\frac{1}{tf_i(x)}$. Remember: $L(\lambda, x) = f_0(x) + \sum_i \lambda_i f_i(x)$.

$L(\lambda, x') \leq \frac{m}{t}$ since $\min_x L(\lambda, x) \leq \frac{m}{t}$

$\mu tf_0(x) - \mu tf_0(x^+) + \sum_i \ln(-\mu t \lambda_i f_i(x^+)) - m \ln \mu$

$\ln(-x) = ln(1 - (1+x)) \leq -(1+x)$

$\leq \mu tf_0(x) - \mu tf_0(x^+) - \sum_i (1 + \lambda_i \mu tf_i(x^+)) - m \ln \mu$

$= \mu tf_0(x) - \mu tf_0(x^+) - \mu t \sum_i \lambda_i f_i(x^+) - m - m \ln \mu$

$= \mu t(f_0(x) - (f_0(x^+) + \sum_i \lambda_i f_i(x^+)) - m - m \ln \mu$

$= \mu t(f_0(x) - L(\lambda, x^+)) - m - m \ln \mu$

$\leq \mu t(\frac{m}{t}) - m - m \ln \mu$

$= m(\mu - 1 - \ln \mu)$

...Central Path Evolution

Old point $x = x(t)$ versus $x^+ = x(\mu t)$?

Minimizing: $F(x) = \mu tf_0(x) + \sum_{i=0}^{n} \ln(-f_i(t))$.

We proved: $F(x) - F(x^+) \leq m(\mu - 1 - \ln \mu)$.

Choose $\mu = 1 + \delta$.

$\ln(1+x) \approx x - x^2/2. \implies \ln(1+\delta) = \delta - \delta^2/2$

$F(x) - F(x^+) = m(1 + \delta - 1 - (\delta - \delta^2/2)) = m(\delta^2/2)$.

Choose $\delta = \frac{1}{\sqrt{m}}$ or $\mu = (1+\delta)$.

$F(x) - F(x^+) = (\delta^2/2)) = \frac{1}{2}$.

Modifying t by factor of $(1 + \frac{1}{\sqrt{m}})$, optimal still close!

t can be arbitrarily large!

The value of the objective function can be gigantic

and change by an enormous amount.

but current point is very close to optimal.

Intuition here?

An attempt at intuition.

$\min \sum_i x_i, x \geq 0.$

optimum is 0.

Central path: $F_t(x) = t \sum_i x_i + \sum_i \ln x_i$

Optimum: $x_i = \frac{1}{t}$.

$t \to \mu t$

New optimum: $x_i^+ = \frac{1}{\mu t}$.

Notice: the change in x is quite small.

Roughly $(\mu - 1)\frac{1}{t} = \frac{1}{\sqrt{m}t}$ where t is large.

Intuitively: new point is very close to old point.

Slightly more generally.

Only one vertex on polytope.

n inequalities, n unkonwns: $\min cx, Ax \geq b$.

Is solution bounded or unbounded?

Alg: Linear equation solve for intersection of n inequalities, check if there is some direction of improvement.

Evolution of central path.

Optimal $x(t)$: $\nabla f_0(x) + \sum_{i=1} \frac{1}{tf_i(x)} \nabla f_i(x) = 0$
$tc = -\sum_i \frac{a_i}{a_i x - b_i}$

$s_i = a_i x - b_i$. "Distance' to constraint.
Recall previous example: $x \geq 0$, the x_i are slack variables.
$s = Ax - b$.

Given solution to $x(t)$ with $b - Ax(t) = s(t)$.
Then $Ax(\mu t) - b = s(t)/\mu$ works.

Since only n inequalities, can just solve to get next point.
Answer is easy too.

More generally.

General $Ax \geq b, \min cx$.

Given solution to $x(t)$ with $b - Ax(t) = s(t)$.
Then $b - Ax(\mu t) = s(t)/\mu$ is optimal: $\mu tc = -\sum_i \frac{a_i}{a_i x - b_i}$

Overdetermined if more than n inequalities, so maybe not possible.

So, need to find solution to: $\mu tc = -\sum_i \frac{a_i}{a_i x - b_i}$

Showed solution is at least close in value to old solution on $F(x)$.

One thing to note:
if you know the optimal vertex (tight constraints).
then you are done.

Idea: close enough to tight constraints. Done.

Close enough to a vertex, can jump to vertex.

Cramer's rule, gives estimate of how close the closest two vertices can be.

Interior Point Method.

Find central point.

Recall: $F(x) = tf_0(x) - \sum_i \log(-f_i(x))$.

Find point: $G(x) = \nabla F(x) = 0$.

Newton: find all zeros of vector valued $G(x)$!

$g_1(x) = \frac{t\partial f_0(x)}{\partial x_1} - \sum_i \frac{\partial f_i(x)}{\partial x_1} \frac{1}{f_i(x)}$

Newton:
$\implies |(x_{n+1} - \alpha)| \leq |\frac{g'(\psi)}{2g''(x_n)}|(x_n - \alpha)^2$.

Recall, distance for x to x^+ is pretty small.
On the order of $1/t$.

What about the ratio? $|\frac{g''(\psi)}{2g'(x_n)}|$

Newton's Method.

$f(x) = \log(a_i x - b_i)$

$s_i = b_i - a_i x$

$f'(x) = \frac{a_i}{s_i}$ $\quad f''(x) = \frac{1}{s_i^2} a_i a_i^T$ $\quad f'''(a) = a_i^{\otimes 3} \frac{1}{s_i^3}$

$f(x + u) = f(x) + f'(x) \cdot u + u^{\otimes 2} \cdot f''(x)$

The minimizer? $-\frac{1}{2}(f''(x))^{-1} f'(x)$.

Self-Concordance: $|f'''(x)| \leq 2f''(x)^{3/2}$.

Newton: $x = x - \frac{f'(x)}{f''(x)}$.
If $f'(x)$ is linear, goes to $f'(x) = 0$.
Scaled by slope, $f''(x)$, of $f'(x)$.

Another Newton Method Analysis:
potential function: $\|f'(x)\| / \|f''(x)\|$.

Idea: if $f''(x)$ does not change, then $f'(x) = 0$.
$f'''(x) \leq f'(x)^{3/2} \to f''(x)$ does not change much.
potential $\|f'(x')\| / \|f''(x')\|$ decreases.

Behavior of log barrier.

What about the ratio? $|\frac{g''(\psi)}{2g'(x_n)}|$

What if $f(x) = \log x$ and recall $g(x) = f'(x)$?

$(\log x)' = 1/x$, $(\log x)'' = -1/x^2$, $(\log x)''' = 2/x^3$.

$|(\log x)'''| = 2|(\log x)''|^{3/2}$.

Thus, this ratio is around $1/x$.

Newton analysis we did: $|\frac{g''(\psi)}{2g'(x_n)}|(x - x^+) < 1$.

Quadratic convergence: ratio is small.

Another Type of IPM strategy.

$\min xc, Ax \geq b$.

$F(x) = tcx - \sum_i \log(a_i x - b_i)$.

$\nabla F(x) = tc - \sum_i \frac{a_i}{s_i}$

Introduce dual variables: λ_i.
Approximate Complementary slackness.
$\lambda_i s_i = \frac{1}{t}$ verse $\lambda_i s_i = 0$
$s = b - Ax$

Predictor-Corrector:
(1) decrease $F(x)$
(2) Fix complementary slackness.

Gives another possibility:
Explicitly maintain primal-dual soluion: (x, s)

Gradient descent and Newton.

minimize $f(x)$.

Gradient descent.

$x = x - \alpha f'(x)$.

Decreases function until gradient changes sign.

If $f''(y) \leq M$ for $y = x - \alpha f'(x)$.

Improve when: $f'(y) > f'(x) - \int_x^y f''(y) dx > 0$.
Also: $f'(y) > f'(x) - M(y - x) > 0$.
When: $(y - x) \leq f'(x)/M$.
set $\alpha = 1/M$.

For linear function: $f'(x)$? Optimum? Is infinitely far.

Newton: $x = x - \frac{f'(x)}{f''(x)}$.

If $f''(x) \geq m$, then $f'(x) = 0$. x: $(x' - x) \leq f'(x)/m$.

Estimate of how far is $f'(x)/f''(x)$. Analysis: the estimate decreases.

so long as $f''(x)$ does not change too much.