An aside.

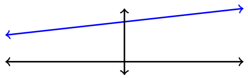
How do you minimize a function?

$argmin_{x\in[a,b]} f(x)$?

Find x?
(1) $f'(x)=0$ and check.
(2) or at the endpoints of an interval.

Calculus.

Linear functions. Derivative is constant, never/always 0.



Unbounded unless restricted to an interval.
Then "at" a vertex in one dimension.
At an endpoint.

Constrained optimization: calculus on an interval.

Vertex solution.

An argument, if not at a vertex can move in a direction.
Keeping current constraints tight.

So do it until you hit another constraint.

Subtle: there may be no vertices.
$\max x_1, x_1 \le 4, x_2 \ge 0$.

There are no "vertices" in the "feasible region."

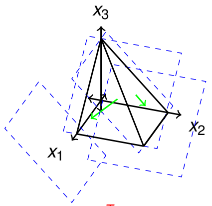
Convex hyperplane separator.

If a point $b \notin P$, for a set P which is convex.
then there is y, s.t., $y^T x > y^T b, \forall x \in P$.

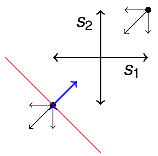
Geometry again.

$Ax=b$, $x \ge 0$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} x = \begin{bmatrix}1\\1\end{bmatrix}$$



Coordinates $s=b-Ax$.
$x \ge 0$ where $s=0$?



Separating Hyperplane.

y where $y^T(b-Ax) < y^T(0) = 0$ for all $x \ge 0$ $\to y^T b < 0$ and $y^T A \ge 0$.
Why? If $y \cdot A^{(i)} < 0$, then take $x_i \to \infty$, $y^T b - y^T Ax \to +\infty$,
Contradiction.

Farkas A: Solution for exactly one of:
(1) $Ax=b, x\ge 0$ or (2) $y^T A \ge 0, y^T b < 0$.

Farkas 2

Farkas A: Solution for exactly one of:
(1) $Ax=b, x\ge 0$
(2) $y^T A \ge 0, y^T b < 0$.

Farkas B: Solution for exactly one of:
(1) $Ax \le b$
(2) $y^T A = 0, y^T b < 0, y \ge 0$.

Strong Duality

(From Goemans notes.)

Primal P $z^* = \min c^T x$
$Ax=b$
$x \ge 0$

Dual D : $w^* = \max b^T y$
$A^T y \le c$

Weak Duality: x, y- feasible P, D: $x^T c \ge b^T y$.

$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T(c - A^T y) \\ &\ge 0 \end{aligned}$$

Strong Duality

P ($Ax = b, \min cx, x \geq 0$): feasible, bounded $\implies z^* = w^*$.

Primal feasible, bounded, minimum value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want y where $\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}$. Let $A' = \begin{pmatrix} A^T \\ -b^T \end{pmatrix}$

Recall Farkas B: Either (1) $A'x' \leq b'$ or (2) $y'^T A' = 0, y'^T b' < 0, y' \geq 0$.

If (1) then done, otherwise (2) $\implies \exists y' = [x, \lambda] \geq 0$.

$$(A \quad -b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \qquad (c^T \quad -z^*) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$. Any feasible $\tilde{x}$ for Primal.
(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.
(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible
$c^T(\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \to -\infty$ as $\mu \to \infty$ Primal unbounded!

Lagrangian Dual.

Find x, subject to

$f_i(x) \leq 0, i = 1, \dots m.$

Remember calculus (constrained optimization.)

Lagrangian: $L(x, \lambda) = \sum_{i=1}^m \lambda_i f_i(x)$

λ_i - Lagrangian multiplier for inequality i, must be positive.

For feasible solution x, $L(x, \lambda)$ is

(A) non-negative in expectation

(B) positive for any λ.

(C) non-positive for any valid λ.

If λ, where $L(x, \lambda)$ is positive for all x

(A) there is no feasible x.

(B) there is no x, λ with $L(x, \lambda) < 0$.

Lagrangian:constrained optimization.

$$\begin{array}{rll} & \min & f(x) \\ \text{subject to} & f_i(x) \leq 0, & i = 1, ..., m \end{array}$$

Lagrangian function:

$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i f_i(x)$

If (primal) x value v
For all $\lambda \geq 0$ with $L(x, \lambda) \leq v$
Maximizing: λ only positive when? $f_i(x) = 0$.

If there is λ with $L(x, \lambda) \geq \alpha$ for all x
Optimum value of program is at least α

Primal problem:
x, that minimizes $L(x, \lambda)$ over all $\lambda \geq 0$.

Dual problem:
λ, that maximizes $L(x, \lambda)$ over all x.

Linear Program.

$\min cx, Ax \geq b$

$$\begin{array}{rll} & \min & c \cdot x \\ \text{subject to} & b_i - a_i \cdot x \leq 0, & i = 1, ..., m \end{array}$$

Lagrangian (Dual):

$L(\lambda, x) = cx + \sum_i \lambda_i (b_i - a_i x_i)$.

or

$L(\lambda, x) = -(\sum_j x_j (a_j \lambda - c_j)) + b\lambda$.

Best λ? Good against every x? Any term $(a_j \lambda - c_j) \neq 0$ is bad.
$\max b \cdot \lambda$ where $a_j \lambda = c_j$.
Why is this good? Every x is the same.

$\max b\lambda, \lambda^T A = c, \lambda \geq 0$
Dual to linear program.

Interior point on the central path.

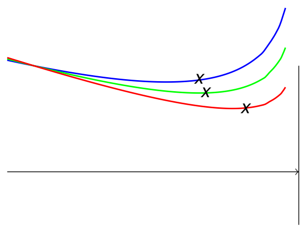
Find x, that minimizes $f_0(x)$ subject to

$f_i(x) \leq 0, i = 1, \dots m.$

Central path:

$\min tf_0(x) - \sum_{i=1} m \ln(-f_i(x))$

The minimizer, $x(t)$, form the **central path.**



The sequence of x's are "central path".

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$ $\nabla f_0(x) + \sum_{i=1} \frac{1}{tf_i(x)} \nabla f_i(x) = 0$

Or, $\nabla f_0(x) = -\sum_{i=1} \frac{\nabla f_i(x)}{tf_i(x)}$ (Opposing force fields.)

Recall, Lagrangian: $L(\lambda, x) = f_0(x) + \sum_i \lambda_i f_i(x)$.

Fix λ, optimize for x^* give valid lower bound on solution.
Optimality Condition.
Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0.$

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Value? $f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$.

Central point $x(t)$ within $\frac{m}{t}$ of optimal!!!!
$L(\lambda, x(t)) \geq f_0(x) - \frac{m}{t} \implies \min_x L(\lambda, x) + \frac{m}{t} \geq f_0(x)$
$\implies OPT + \frac{m}{t} \geq f_0(x)$

Central path.

$\min_x f_0(x), f_i(x) \leq 0.$

$\min_x tf_0(x) - \sum_{i>0} \ln(-f_i(x))$

Optimal: $x(t)$ is feasible.

$f_0(x(t)) \geq OPT - \frac{m}{t}$

Algorithm: take $t \to \infty$.

Finding $x(t)$?

Next.