Today

Nash's Theorem

Strategic Games.

N players.

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Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

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Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

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Nash Equilibrium:

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Nash Equilibrium:
neither player has incentive to change strategy.

Proving Nash.

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What is x? A vector of vectors: vector i is length m_i.

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What is $x_{-i}; z$?

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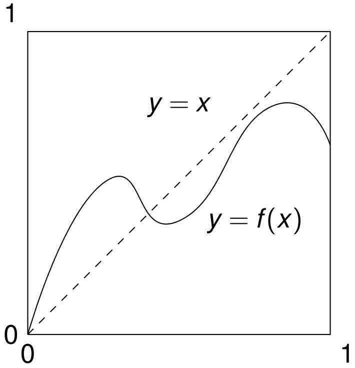
Theorem: There is a Nash Equilibrium.

Brouwer Fixed Point Theorem.

Theorem: Every continuous function from a closed compact convex (c.c.c.) set to itself has a fixed point.



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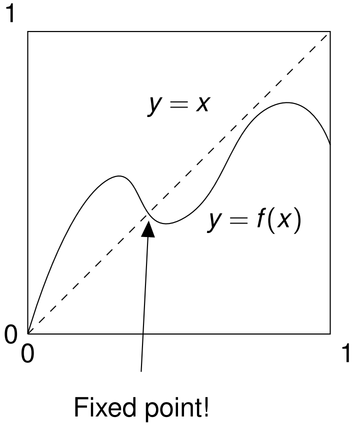

What is the closed convex set here?

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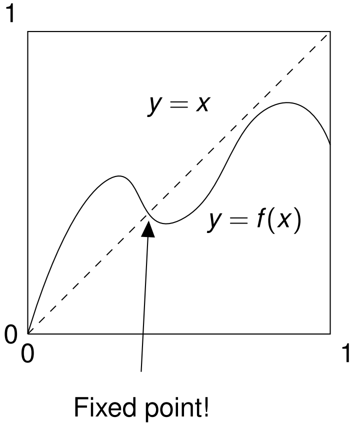

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The unit square?

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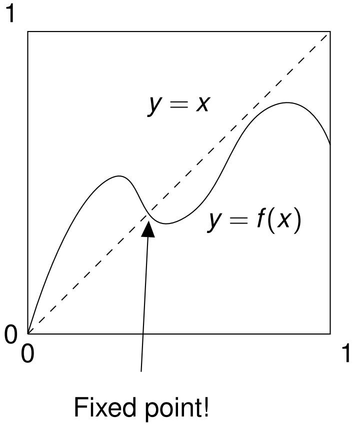

What is the closed convex set here?
The unit square? Or the unit interval?

Brouwer implies Nash.

The set of mixed strategies x is closed convex set.

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Unique minimum as it is quadratic.

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$\phi(\cdot)$ is continuous on the closed convex set.

Brouwer: Has a fixed point: $\phi(\hat{z}) = \hat{z}$.

Quick Almost Irrelevant Question.

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Change coordinates proportional to utility differences.

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Tradeoffs squared penalty function against benefit in utility.

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Technically: need to project back to feasible set.

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  Distribution for each player.

Fixed Point is Nash.

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$\implies$ that the limit point has $f(x)_i = x_i$ for all i.

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For any $n+1$-dimensional simplex and a subdivion into smaller simplices.

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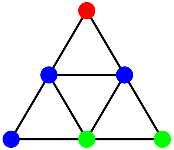

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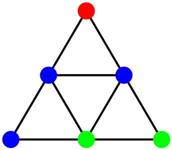

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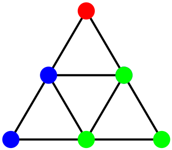

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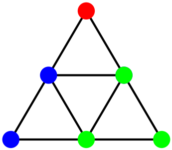

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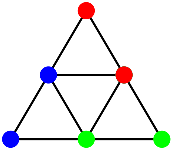

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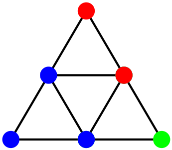

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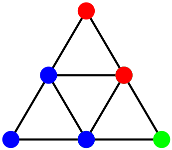

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By induction!

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One dimension:

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One dimension: Subdivision of $[0, 1]$.

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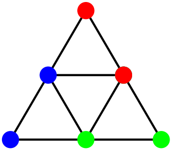

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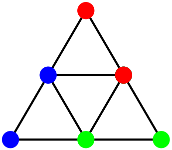
Consider (r, g) edges.



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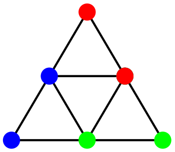
Separates two regions.



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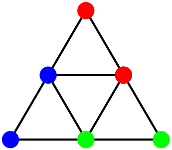

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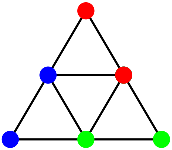

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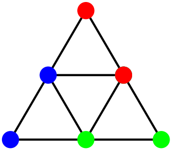

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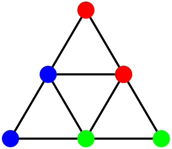

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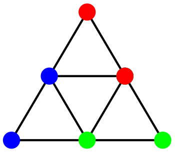

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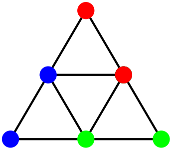

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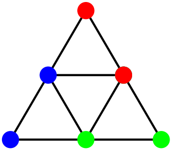

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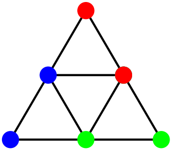

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Must be odd number!

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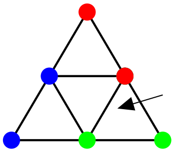

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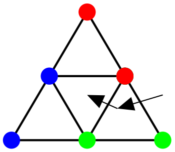

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Q: counts "almost rainbow" cells; has $\{1,\ldots,n\}$.
Note: exactly one color in $\{1,\ldots,n\}$ used twice.

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R: counts "rainbow" cells; has all $n+1$ colors.

Claim: there is an odd number of rainbow cells.

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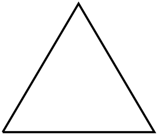

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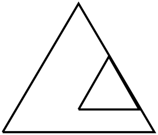

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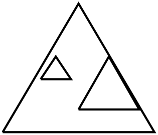

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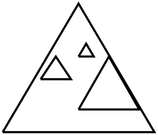

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Area is $\sum_e d(e)$.

Width is cut-size. Rate of growth at each $d(S_\ell)$.

Region Growing: Warmup.

Length $\times$ Width = Area.

Width $\leq$ Area/Length at some point in region.

Lemma: If $\exists i,j, d(i,j) \geq \Delta$, then $\exists S$ with $E(S,\overline{S}) \leq \frac{\sum_e d(e)}{\Delta}$.

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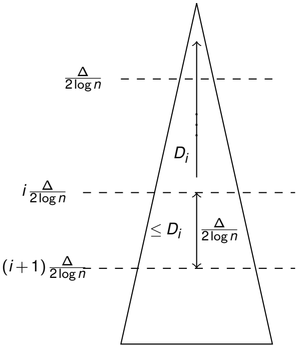
Again with a picture.



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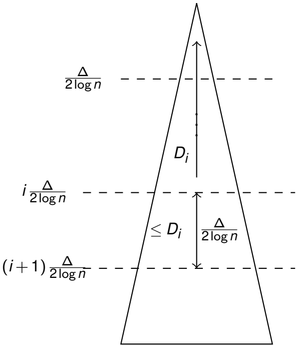

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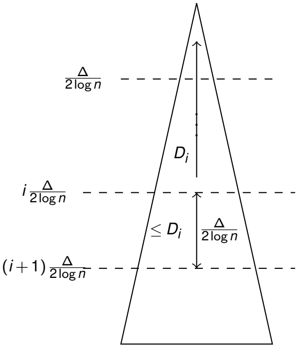

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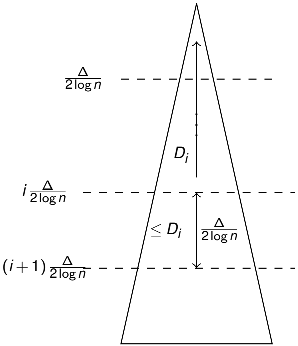

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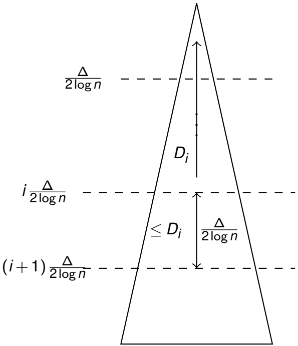

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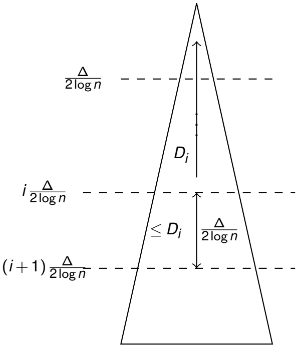

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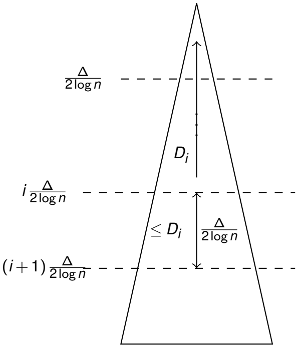

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Finds cut of sparsity $O(\log n/\alpha)$.

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Do some averaging to get real result.

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Low diameter decomposition.

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Procedure produces cluster of Diameter $O(\Delta)$.

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$O(\frac{\log n}{\Delta})$ fraction of edges in between.

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Repeat until every vertex in a cluster.

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Repeat until every vertex in a cluster.

Produces:

Decompositon into low-diameter clusters: $O(\Delta)$.

A structure.

Low diameter decomposition.

Procedure produces cluster of Diameter $O(\Delta)$.

$O(\frac{\log n}{\Delta})$ fraction of edges in between.

Repeat until every vertex in a cluster.

Produces:

Decompositon into low-diameter clusters: $O(\Delta)$.

Edges between “Small”:

A structure.

Low diameter decomposition.

Procedure produces cluster of Diameter $O(\Delta)$.

$O(\frac{\log n}{\Delta})$ fraction of edges in between.

Repeat until every vertex in a cluster.

Produces:

Decompositon into low-diameter clusters: $O(\Delta)$.

Edges between "Small": $\tilde{O}(\frac{1}{\Delta})$.

A structure.

Low diameter decomposition.

Procedure produces cluster of Diameter $O(\Delta)$.

$O(\frac{\log n}{\Delta})$ fraction of edges in between.

Repeat until every vertex in a cluster.

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Decompositon into low-diameter clusters: $O(\Delta)$.

Edges between “Small”: $\tilde{O}(\frac{1}{\Delta})$.

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Low diameter decomposition.

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$O(\frac{\log n}{\Delta})$ fraction of edges in between.

Repeat until every vertex in a cluster.

Produces:

Decompositon into low-diameter clusters: $O(\Delta)$.

Edges between "Small": $\tilde{O}(\frac{1}{\Delta})$.

$\tilde{O}(\cdot)$ hides log factors.