Today

Nash's Theorem

Strategic Games.

N players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(.1.1)

What is the best thing for the players to do?

Both cooperate. Payoff $(3,3)$.

If player 1 wants to do better, what does she do?

Defects! Payoff $(5,0)$

What does player 2 do now?

Defects! Payoff $(.1,.1)$.

Stable now!

Nash Equilibrium:
neither player has incentive to change strategy.

Proving Nash.

n players.

Player i has strategy set $\{1, \ldots, m_i\}$.

Payoff function for player i: $u_i(s_1, \ldots, s_n)$ (e.g., $\in \Re^n$).

Mixed strategy for player i: x_i is vector over strategy set.

Nash Equilibrium: $x = (x_1, \ldots, x_N)$ where

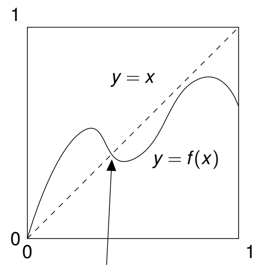
$$\forall i \forall x_i', u_i(x_{-i}; x_i') \leq u_i(x). \qquad (1)$$

What is x? A vector of vectors: vector i is length m_i.
What is $x_{-i}; z$? x with x_i replaced by z.
What (1) does say? No new strategy for player i that is better!

Theorem: There is a Nash Equilibrium.

Brouwer Fixed Point Theorem.

Theorem: Every continuous function from a closed compact convex (c.c.c.) set to itself has a fixed point.



Fixed point!

What is the closed convex set here?
The unit square? Or the unit interval?

Brouwer implies Nash.

The set of mixed strategies x is closed convex set.

That is, $x = (x_1, \ldots, x_n)$ where $|x_i|_1 = 1$.

$\alpha x' + (1-\alpha)x''$ is a mixed strategy.

Define $\phi(x_1, \ldots, x_n) = (z_1, \ldots, z_n)$

where $z_i = \arg\max_{z_i'} \left[u_i(x_{-i;z_i'}) - \|z_i - x_i\|_2^2 \right]$.

Unique minimum as it is quadratic.
z_i is continuous in x.
Mixed strategy utilities is polynomial of entries of x
with coefficients being payoffs in game matrix.

$\phi(\cdot)$ is continuous on the closed convex set.

Brouwer: Has a fixed point: $\phi(\hat{z}) = \hat{z}$.

Quick Almost Irrelevant Question.

Define $\phi(x_1,\dots,x_n) = (z_1,\dots,z_n)$

where $z_i = \arg\max_{z_i'} \left[u_i(x_{-i;z_i'}) - \|z_i - x_i\|_2^2 \right]$.

Question: which way will it go?
Some pure strategy is (tied for) best response.
Which way will it go?
Change coordinates proportional to utility differences.

Tradeoffs squared penalty function against benefit in utility.

Looks like a gradient.

This is (another) property of the quadratic "regularizer."

Technically: need to project back to feasible set.
Distribution for each player.

Fixed Point is Nash.

$\phi(x_1,\dots,x_n) = (z_1,\dots,z_n)$ where
$z_i = \arg\max_{z_i'} \left[u_i(x_{-i;z_i'}) - \|z_i - x_i\|_2^2 \right]$.

Fixed point: $\phi(\hat{z}) = \hat{z}$

If $\hat{z}$ not Nash, there is i, y_i where

$u_i(\hat{z}_{-i}; y_i) > u_i(\hat{z}) + \delta$.

Consider $\hat{y}_i = (1-\alpha)\hat{z}_i + \alpha(y_i - \hat{z}_i)$.
$(1-\alpha)u_i(\hat{z_{-i}}; \hat{y}_i) + \|\hat{z}_i - \hat{y}_i\|^2$?

$$(1-\alpha)u_i(\hat{z}) + \alpha(u_i(\hat{z}) + \delta) - \alpha^2\|\hat{z}_i - y_i\|^2$$
$$= u_i(\hat{z}) + \alpha\delta - \alpha^2\|y_i - \hat{z}_i\|^2 > u_i(\hat{z}).$$

The last inequality true when $\alpha < \frac{\delta}{\|y_i - z_i\|^2}$.

Thus, $\hat{z}$ not a fixed point!

Thus, fixed point is Nash. □

Proof of Brouwer: outline.

Sperner: any subdivision of a simplex and "proper" coloring of its vertices
$\implies$ a simplex in subdivision which is multicolored.

Given a function, $f(\cdot)$, on the simplex.
Take a sequence of subdivisons, define colorings using $f(\cdot)$.
Multicolored simplices have property that $f(x)_i \leq x_i$ at vertex i.

$\implies$ that the limit point has $f(x)_i = x_i$ for all i.

Sperner's Lemma

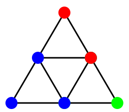
For any $n+1$-dimensional simplex and a subdivion into smaller simplices.

All vertices are colored $\{1,\dots,n+1\}$.

The coloring is proper if the extremal vertices are differently colored.

Each face only contains the colors of the incident corners.

Lemma: There exist a simplex that has all the colors.



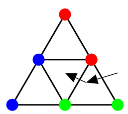
Oops.
Where is multicolored?
Where is multicolored? And now?

By induction!

Proof of Sperner's.

One dimension: Subdivision of $[0,1]$.

Endpoints colored differently.
Odd number of multicolored edges.



Two dimensions.
Consider (r,g) edges.
Separates two regions.
Dual edge connects regions with r on right.
Exterior region has excess out-degree:
one more (r,g) than (g,r).
There exist a region with excess in-degree.

(r,g,r) triangle has in-degree=out-degree.
(g,r,g) triangle has in-degree=out-degree.

Must be (r,g,b) triangle.
Must be odd number!

$n+1$-dimensional Sperner.

R: counts "rainbow" cells; has all $n+1$ colors.

Claim: there is an odd number of rainbow cells.

Q: counts "almost rainbow" cells; has $\{1,\dots,n\}$.
Note: exactly one color in $\{1,\dots,n\}$ used twice.

Rainbow face: $n-1$-dimensional, vertices colored with $\{1,\dots,n\}$.
X: number of boundary rainbow faces.
Y: number of internal rainbow faces.

Number of Rainbow Face to Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces only on one face of big simplex: $\{1,\dots,n\}$
Induction $\implies$ Odd number of rainbow faces.
$\implies$ X is odd $\implies$ $X + 2Y$ is odd $\implies$ $R + 2Q$ is odd.
R is odd. □

Sperner to Brouwer

Consider simplex:S.

Closed compact sets can be mapped to simplex. Somehow?

Let $f : S \to S$. (Note from n-dimensions to n-dimensions.)

Infinite sequence of subdivisions: $\mathscr{S}_1, \mathscr{S}_2, \ldots$

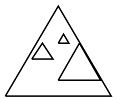
$\mathscr{S}_j$ is subdivision of $\mathscr{S}_{j-1}$. Size of cell $\to 0$ as $j \to \infty$.

A coloring of $\mathscr{S}_j$. Recall $\sum_i x_i = 1$ in simplex.
Big simplex vertices $e_j = (0,0,\ldots,1,\ldots,0)$ get j.

For a vertex at x.
Assign smallest i with $f(x)_i < x_i$.
Exists? Yes. Since $\sum_i f(x)_i = \sum_i x_i$. And not fixed point.

Proper Coloring? Simplex face is at $x_j = 0$ for opposite j.
Thus $f(x)_j$ cannot be smaller and is not colored j.

Rainbow cell, in $\mathscr{S}_j$ with vertices $x^{j,1}, \ldots, x^{j,n+1}$.



Rainbow Cells to Brower.

Rainbow cell, in $\mathscr{S}_j$ with vertices $x_j^{j,1}, \ldots, x_j^{j,n+1}$.

Each set of points $x^{j,k}$ is an infinite set in S.

$\to$ This is a convergent subsequence $\to$ has limit point.
$\to$ All have same limit point as they get closer together.
x^* is (common) limit point.

$f(x^*)$ not fixed point $\implies$ $f(x^*)_i > x_i^*$ for some i. (Since $\sum_i x_i^* = 1$).

But $f(x^{j,i})_i < x_i^{j,i}$ for all j and i and
$\lim_{j\to\infty} x^{j,i} = x^*$.

Thus, $f(x^*)_i \leq x_i^*$ by continuity. Contradiction. □

Computing Nash Equilibrium.

PPAD - "Polynomial Parity Argument on Directed Graphs."

"Graph with unbalanced node (indegree $\neq$ outdegree) has another."

Exponentially large graph with vertex set $\{0,1\}^n$.

Circuit given name of graph finds previous, $P(v)$, and next, $N(v)$.

Sperner: local information gives neighbor.

END OF THE LINE. Given circuits P and N as above, if O^n is unbalanced node in the graph, find another unbalanced node.

PPAD is search problems poly-time reducibile to END OF LINE.

NASH $\to$ BROUWER $\to$ SPERNER $\to$ END OF LINE $\in$ PPAD.

Other classes.

PPA: "If undirected graph has a node of odd degree, it has another.

PLS: "Every directed acyclic graph must have a sink."

PPP:
"If a function maps n elements to $n-1$ elements, there is a collision."

All exist: not NP!!! Answer is yes. How to find quickly?

Reduction:
END OF LINE $\to$ Piecewise Linear Brouwer $\to$ $3D$–Sperner$\to$ Nash.

Uh oh. Nash is PPAD-complete.

Who invented? Papadimitriou PapaD and PPAD. Perfect together!

Sparsest cut approximation

Sparsity of a cut S: $\frac{E(S,\overline{S})}{|S|\times|\overline{S}|}$.

Similar to $h(G)$ from Cheeger: $\frac{E(S,\overline{S})}{d|S|}$.
Factor of two approximations of each other.

Sparsity of a graph is $\min_S \frac{E(S,\overline{S})}{|S|\times|\overline{S}|}$.

What is the sparsity of cuts in a the complete graph, K_n?

for $|S| = k$, sparsity is $\frac{(k)(n-k)}{\binom{n}{2}}$.
minimum at $k = n/2$, and is $\approx 1/4$.

Sparsity, complete graph and embeddings.

Graph, $G = (V,E)$, with sparsity α.
Minimum congestion to embed complete graph?
Cut: $(S,\overline{S})$ has $|E(S,S)| = \alpha|S| \times |\overline{S}|$
Complete $|S| \times |\overline{S}|$ routed over $|E(S,\overline{S})|$ edges.
Average edge has to carry $|S| \times |\overline{S}|/|E(S,\overline{S})|$ paths.
Congestion is $\geq 1/\alpha$.

Linear program gives lower bound on α.

Multicommodity flow (path routing) computes minimum congestion?

Is this an approximation to the minimum sparsity?

Theorem: the complete graph can be routed with $O(\log n/\alpha)$ congestion.

Gives upper bound on α. $O(\log n)$ approximation.

Dual linear program.

Toll problem.

Assign $d(e)$ for edge $e \in E$ with $\sum_e d(e) = 1$, to maximize $\sum_{i,j} d(i,j)$.

Consider cut: $(S, \overline{S})$,
assign $d(e) = 1/|E(S, \overline{S}|$ for $e \in E(S, \overline{S})$.

$\sum_{i,j} d(i,j) = \frac{|S| \times |\overline{S}|}{E(S, \overline{S})} = \frac{1}{\alpha}$.

Theorem: The value of the dual is $O(\frac{\log n}{\alpha})$.

Region Growing: Warmup.

Length $\times$ Width = Area.
Width $\leq$ Area/Length at some point in region.

Lemma: If $\exists i,j, d(i,j) \geq \Delta$, then $\exists S$ with $E(S, \overline{S}) \leq \frac{\sum_e d(e)}{\Delta}$.

Length is $d(i,j)$ or Δ.
$d(i,x)$ is distance to x.
S_ℓ with $d(i,x) \leq \ell$ is a cut.
Think Djikstra's or "Breadth First Search".
Technically fraction of edges inside S_ℓ.
Area is $\sum_e d(e)$.
Width is cut-size. Rate of growth at each $d(S_\ell)$.

Or with natural numbers:
Breadth first search tree of depth D.
Each level is a cut.
There exists a cut of size m/D.
Cuz:
Length (depth) times width (cut size) is area (number of edges.)

Problem is that it is not "balanced."

Region Growing.

Lemma: $\exists x, y\, d(x,y) \geq \Delta \implies$ cut, S, where $|E(S, \overline{S})| \leq O(\frac{d(S) \log n}{\Delta})$.

Extend, $d(\cdot)$ to vertices: $d(v) = \frac{\sum_e d(e)}{n}$.

Let S_ℓ be v where $d(x,v) \leq \ell$..
Define $D(x, \ell)$ to be the sum of:
(1) $d(v)$ for $v \in S_\ell$.
(2) For $e = (u,v)$, $d(e)$ where $u, v \in S_\ell$
(3) For $e = (u,v)$, $u \in S_\ell, v \notin S_\ell$, $\ell - d(u)$.

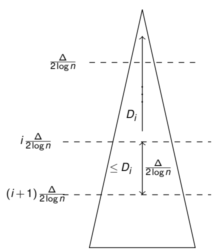
W.L.O.G. $D(x, \Delta/2) \leq \frac{2\sum_e d(e)}{2}$. Ball contains $\leq$ half the weight.

Let $D_i = D(x, i\frac{\Delta}{(2 \log n)})$.

Claim: Exists i such that $D_{i+1} \leq 2D_i$.
Proof: Can't double more than $\log n$ times.

Claim: Exists a cut, S, where $d(S_\ell) \geq D_i$ and $E(S, |S|) \leq \frac{2D_i \log n}{\Delta}$
Proof: Interval i: has length $\frac{\Delta}{2 \log n}$ and area $\leq D_i$
Width $\frac{2D_i \log n}{\Delta}$. □

Again with a picture.



Claim:
Exists cut, S, such that $E(S, \bar{(}S)) \leq \frac{2D(S) \log n}{\Delta}$

Call $d(e)$ weight.
Exists some i, where weight, $D_{i+1} - D_i \leq D_i$.
Weight is like Area: Cut-size $\times$ length.
Cut-size $\leq$ Area/length $= \frac{D_i}{\Delta/2 \log n} = \frac{2D_i}{\log n}$

Done.

Approximation Algorithm

Claim: $\frac{E(S, \overline{S})}{D(S)} \leq \frac{2 \log n}{\Delta}$.

Linear program value: $\sum_{i,j} d(i,j) \geq \alpha = \min_S \frac{E(S,S)}{|S||\overline{S}|}$.

There exists vertex i, j, where $d(i,j) = \Delta => \alpha/n^2$.

$\implies \frac{E(S, \overline{S})}{n^2 D(S)} \leq \frac{2 \log n}{\alpha}$.

Scenario: $D(S) = \Omega(1)$ and $|S| = \Omega(n)$.
Finds cut of sparsity $O(\log n/\alpha)$. Optimal is $\geq \frac{1}{\alpha}$.

$O(\log n)$ approximation.

Do some averaging to get real result.

A structure.

Low diameter decomposition.

Procedure produces cluster of Diameter $O(\Delta)$.

$O(\frac{\log n}{\Delta})$ fraction of edges in between.

Repeat until every vertex in a cluster.
Produces:
Decompositon into low-diameter clusters: $O(\Delta)$.
Edges between "Small": $\tilde{O}(\frac{1}{\Delta})$.

$\tilde{O}(\cdot)$ hides log factors.