Rules for School...

or..."Rules for taking duals"

Standard:

$Ax \leq b, \max cx, x \geq 0 \leftrightarrow y^T A \geq c, \min by, y \geq 0.$

$\min \leftrightarrow \max$

$\geq \leftrightarrow \leq$

"inequalities" $\leftrightarrow$ "nonnegative variables"

"nonnegative variables" $\leftrightarrow$ "inequalities"

One more useful trick: Equality constraints.

"equalities" $\leftrightarrow$ "unrestricted variables."

Maximum Weight Matching.

Bipartite Graph $G = (V,E)$, $w : E \to Z$.
Find maximum weight perfect matching.
Solution: x_e indicates whether edge e is in matching.

$$\max \sum_e w_e x_e$$
$$\forall v: \sum_{e=(u,v)} x_e = 1 \qquad p_v$$
$$x_e \geq 0$$

Dual.
Variable for each constraint. p_v unrestricted.
Constraint for each variable. Edge e, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$
$$\forall e = (u,v): \quad (p_u + p_v) \geq w_e$$

Weak duality? Price function upper bounds matching.
$\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} (p_u + p_v) \leq \sum_v p_u$.

Strong Duality? Same value solutions. Hungarian algorithm !!!

Complementary Slackness.

$$\max \sum_e w_e x_e$$
$$\forall v: \sum_{e=(u,v)} x_e = 1 \qquad p_v$$
$$x_e \geq 0$$

Dual:

$$\min \sum_v p_v$$
$$\forall e = (u,v): \quad p_u + p_v \geq w_e$$

Complementary slackness:
Only match on tight edges.
Nonzero p_u on matched u.

Multicommodity Flow.

Given $G = (V,E)$, and capacity function $c : E \to Z$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $d_1, \ldots, d_k$.
Route D_i flow for each s_i, t_i pair,
so every edge has $\leq \mu c(e)$ flow
and minimize μ.

variables: f_p flow on path p.
P_i -set of paths with endpoints s_i, t_i.

$$\min \mu$$
$$\forall e: \sum_{p \ni e} f_p \leq \mu c_e$$
$$\forall i: \sum_{p \in P_i} f_p = D_i$$
$$f_p \geq 0$$

Take the dual.

$$\min \mu$$
$$\forall e: \sum_{p \ni e} f_p \leq \mu c_e$$
$$\forall i: \sum_{p \in P_i} f_p = D_i$$
$$f_p \geq 0$$

Modify to make it $\geq$, which "goes with" min.
And only constants on right hand side.

$$\min \mu$$
$$\forall e: \mu c_e - \sum_{p \ni e} f_p \geq 0$$
$$\forall i: \sum_{p \in P_i} f_p = D_i$$
$$f_p \geq 0$$

Dual.

$$\min \mu$$
$$\forall e: \mu c_e - \sum_{p \ni e} f_p \geq 0 \qquad d_e$$
$$\forall i: \sum_{p \in P_i} f_p = D_i \qquad d_i$$

Introduce variable for each constraint.
Introduce constraint for each var:
$\mu \to \sum_e c_e d_e = 1$. $f_p \to \forall p \in P_i \; d_i - \sum_{e \in p} d_e \leq 0$.
Objective: right hand sides. $\max \sum_i D_i d_i$

$$\max \sum_i D_i d_i$$
$$\forall p \in P_i: d_i \leq \sum_{e \in p} d(e)$$
$$\sum_e c_e d_e = 1$$

d_i - shortest s_i, t_i path length. Toll problem!
Weak duality: toll lower bounds routing.
Strong Duality. Tight lower bound. First lecture. Or Experts.
Complementary Slackness: only route on shortest paths
only have toll on congested edges.

Exponential size.

Multicommodity flow.

$$\min \mu$$
$$\forall e: \mu c_e - \sum_{p \ni e} f_p \geq 0$$
$$\forall i: \sum_{p \in P_i} f_p = d_i$$
$$f_p \geq 0$$

Dual is.

$$\max \sum_i D_i d_i$$
$$\forall p \in P_i: d_i \leq \sum_{e \in p} d(e)$$

Exponential sized programs?

Answer 1: We solved anyway!

Answer 2: Ellipsoid algorithm.
Find violated constraint $\rightarrow$ poly time algorithm.

Answer 3: there is polynomial sized formulation.
Question: what is it?

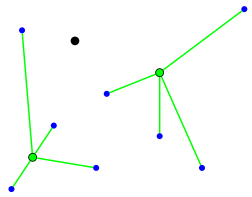
Facility location

Set of facilities: F, opening cost f_i for facility i

Set of clients: D.

d_{ij} - distance between i and j.
(notation abuse: clients/facility confusion.)

Triangle inequality: $d_{ij} \leq d_{ik} + d_{kj}$.

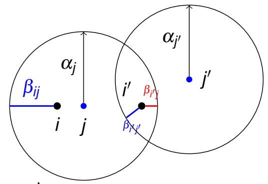


LP and Dual. Interpretation?

$$\min \sum_{i \in F} y_i f_i + \sum_{i \in F, j \in D} x_{ij} d_{ij}$$
$$\forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1$$
$$\forall i \in F, j \in D \quad x_{ij} \leq y_i,$$
$$x_{ij}, y_i \geq 0$$

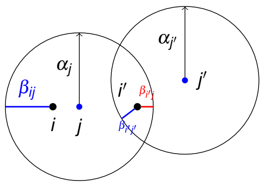
$$\max \sum_j \alpha_j$$
$$\forall i \in F \quad \sum_{j \in D} \beta_{ij} \leq f_i$$
$$\forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij} \quad x_{ij}$$
$$\alpha_j, \beta_{ij} \geq 0$$

α_j charge to client.
maximize price for client to connect!
Objective: $\sum_j \alpha_j$ total payment.
Client j travels or pays to open facility i.
Costs client d_{ij} to get to there.
Savings is $\alpha_j - d_{ij}$.
Willing to pay $\beta_{ij} = \alpha_j - d_{ij}$.



Total payment to facility i at most f_i before opening.
Complementary slackness: $x_{ij} \geq 0$ if and only if $\alpha_j \geq d_{ij}$.
only assign client to "paid to" facilities.

Use Dual.



1. Find solution to primal, (x, y), and dual, (α, β).
2. For smallest (remaining) α_j,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility i in N_j.
   Every client j' with $N_{j'} \cap N_j \neq \emptyset$ assigned to i.
   ("Balls" overlap.)
3. Removed assigned clients, goto 2.

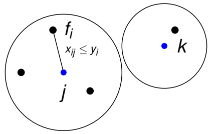
Choose facilities to cover all clients.
Use "balls" of clients to pick which facilities.

Integral facility cost at most LP facility cost.

Claim: Total facility cost is at most $\sum_i f_i y_i$.

Note: Recall LP minimized: $\sum_i y_i f_i + x_{ij} d_{ij}$.

2. For smallest (remaining) α_j,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility i in N_j.
   Every client j' with $N_{j'} \cap N_j \neq \emptyset$ assigned to i.



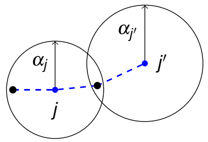
Proof: Step 2 picks client j.
$f_{\min}$ - min cost facility in N_j
$f_{\min} \leq f_{\min} \cdot \sum_{i \in N_j} x_{ij} \leq f_{\min} \sum_{i \in N_j} y_i \leq \sum_{i \in N_j} y_i f_i$.
For $k \neq j$ used in Step 2.
$N_j \cap N_k = \emptyset$ for j and k in step 2.
$\rightarrow$ Any facility in ≤ 1 sum from step 2.
$\rightarrow$ total step 2 facility cost is $\leq \sum_i y_i f_i$.

Connection Cost.

2. For smallest (remaining) α_j,
   (a) Let $N_j = \{i : x_{ij} > 0\}$.
   (b) Open cheapest facility i in N_j.
   Every client j' with $N_{j'} \cap N_j \neq \emptyset$ assigned to i.

   Recall: Dual maximizes: $\sum_j \alpha_j$

Client j is directly connected. Clients j' are indirectly connected.



Connection Cost of j: $\leq \alpha_j$.
Connection Cost of j':
$\leq \alpha_{j'} + \alpha_j + \alpha_j \leq 3\alpha_{j'}$.
since $\alpha_j \leq \alpha_{j'}$

Total connection cost:
at most $3 \sum_j \alpha_j \leq 3$ times Dual OPT.

Previous Slide: Facility cost:
$\leq$ primal "facility" cost $\leq$ Primal OPT.

Total Cost: 4 OPT.

Twist on randomized rounding.

Client j: $\sum_i x_{ij} = 1$, $x_{ij} \geq 0$.
Probability distribution! $\rightarrow$ Choose from distribution, x_{ij}, in step 2.

Expected opening cost: (note: larger than $f_{\min}$.)
$\sum_{i \in N_j} x_{ij} f_i \leq \sum_{i \in N_j} y_i f_i$.
and separate balls implies total $\leq \sum_i y_i f_i$.

$D_j = \sum_i x_{ij} d_{ij}$ Expected connection cost of primal for j.

Expected connection cost j' $\alpha_j + \alpha_{j'} + D_j$.

In step 2: pick in increasing order of $\alpha_j + D_j$.

$\rightarrow$ Expected cost is $\leq (2\alpha_{j'} + D_{j'})$.
Connection cost: $2\sum_j \alpha_j + \sum_j D_j$.
$2OPT(D)$ plus connection cost of primal.

Total expected cost:
Facility cost is at most facility cost of primal.
Connection cost at most $2OPT$ + connection cost of prmal.
$\rightarrow$ at most $3OPT$.

Primal dual algorithm.

1. Feasible integer solution.
2. Feasible dual solution.
3. Cost of integer solution $\leq \alpha$ times dual value.

Just did it. Used linear program. Faster?

Typically. (If dual is maximization.)
Begin with feasible dual.
Raise dual variables until tight constraint.
Set corresponding primal variable to an integer.

Recall Dual:

$$\max \sum_j \alpha_j$$
$$\forall i \in F \quad \sum_{j \in D} \beta_{ij} \leq f_i$$
$$\forall i \in F, j \in D \quad \alpha_j - \beta_{ij} \leq d_{ij}$$
$$\alpha_j, \beta_{ij} \leq 0$$

Facility location primal dual.

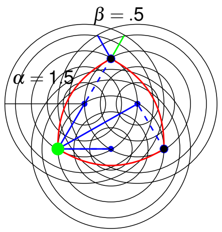
Phase 1: 1. Initially $\alpha_j, \beta_{ij} = 0$.
2. Raise α_j for every (unconnected) client.
When $\alpha_j = d_{ij}$ for some i
raise β_{ij} at same rate Why? Dual: $\alpha_j - \beta_{ij} \leq d_{ij}$.
Intuition:Paying β_{ij} to open i.
Stop when $\sum_i \beta_{ij} = f_i$.
Why? Dual: $\sum_i \beta_{ij} \leq f_i$
Intuition: facility paid for.
Temporarily open i.
Connect all tight ji clients j to i.

3. Continue until all clients connected.

Phase 2:
Make "edge" between two facilities if paid by a common client.
Permanently open an independent set of facilities in graph.

For client j, connected facility i is opened. Good.
Connected facility not open
$\rightarrow$ exists client j' paid i and connected to open facility.
Connect j to j''s open facility.

Example.



Constraints for dual.
$\sum_j \beta_{ij} \leq f_i$
$\alpha_i - \beta_{ij} \leq d_{ij}$.
Grow α_j.
$\alpha_j = d_{ij}$!
Tight constraint: $\alpha_j - \beta_{ij} \leq d_{ij}$.
Grow β_{ij} (and α_j).
$\sum_j \beta_{ij} = f_i$ for all facilities.
Tight: $\sum_j \beta_{ij} \leq f_i$
LP Cost: $\sum_j \alpha_j = 4.5$

Temporarily open all facilities.
Assign Clients to "paid to" open facility.
Connect facilities with common client.
Open independent set.
Connect to "killer" client's facility.
Cost: 1 + 3.7 = 4.7.
A bit more than the LP cost.

Analysis

Claim: Client only pays one facility.

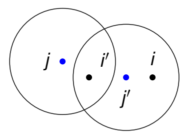
Independent set of facilities.

Claim: S_i - directly connected clients to open facility i.
$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j$.

Proof:
$f_i = \sum_{j \in S_i} \beta_{ij} = \sum_{j \in S_i} \alpha_j - d_{ij}$.
Since directly connected: $\beta_{ij} = \alpha_j - d_{ij}$. □

Analysis.

Claim: Client j is indirectly connected to i
$\rightarrow d_{ij} \leq 3\alpha_j$.



Directly connected to (temp open) i'
has common client j' with some facility i.
client j' has $\alpha_{j'} \geq d_{ij'}$ and $\alpha_j \geq d_{i'j'}$.
When i' opens, stops both α_j and α'_j.
$\alpha_{j'}$ stopped no later (..maybe earlier..)
$\alpha_{j'} \leq \alpha_j$.
Total distance from j to j'.
$d_{ji'} + d_{i'j'} + d_{j'i} \leq 3\alpha_j$ □

Putting it together!

Claim: Client only pays one facility.
Claim: S_i - directly connected clients to open facility i.
$f_i + \sum_{j \in S_i} d_{ij} \leq \sum_j \alpha_j$.
Claim: Client j is indirectly connected to i
$\rightarrow d_{ij} \leq 3\alpha_j$.

Total Cost:
direct clients dual (α_j) pays for facility and own connections.
plus no more than 3 times indirect client dual.
Total Cost: 3 times dual.
feasible dual upper bounds fractional (and integer) primal.

3 OPT.

Fast! Cheap! Safe!