Today.

Markov chain mixing analysis.

Perhaps: reducing the use of randomness.

Sampling.

Sampling: Random element of subset $S \subset \{0,1\}^n$ or $\{0,\ldots,k\}^k$.

Related Problem: Approximate $|S|$ within factor of $1+\varepsilon$.

Random walk to do both for some interesting sets S.

Convex Bodies.

$S \subset [k]^n$ is grid points inside Convex Body.

Ex: Numerically integrate convex function in d dimensions.
Compute $\sum_i v_i Vol(f(x) > v_i)$ where $v_i = i\delta$.

Example: P defined by set of linear inequalities.
Or other “membership oracle” for P

S is set of grid points inside Convex Body.
Grid points that satisfy linear inequalities.
or “other” membership oracle.
Choose a uniformly random elt?
Easy to choose randomly from $[k]^n$ which is big.

For convex body?
Choose random point in $[k]^n$ and check if in P.
Works.
But P could be exponentially small compared to $|[k]^n|$.
Takes a long time to even find a point in P.

Convex Body Graph.

$S \subset [k]^n$ is set of grid points inside Convex Body.

Sample Space: S.

Graph on grid points inside P or on Sample Space.

One neighbor in each direction for each dimension
(if neighbor is inside P.)
Degree: $2d$.

How big is graph? Big!
So big it ..it INSERT JOKE HERE.
$O(k^n)$ if coordinates in $[k]$.
That's a big graph!

How to find a random node?

Start at a grid point, and take a (random) walk.

When close to uniform distribution...have a sample point.

How long does this take? More later.
But remember power method...which finds first eigenvector.

Spanning Trees.

Problem: How many?

Another Problem: find a random one.

Algorithm:
Start with spanning tree.
Repeat:
Swap a random nontree edge with a random tree edge.

How long?

Sample space graph (BIG GRAPH) of spanning trees.
Node for each tree.
Neighboring trees differ in two edges.

Algorithm is random walk on BIG GRAPH (sample space graph.)

Spin systems.

Each element of S may have associated weight.

Sample element proportional to weight.

Example?
2 or 3 dimensional grid of particles.
Particle State ± 1. System State $\{-1,+1\}^n$.

Energy on local interactions: $E = \sum_{(i,j)} -\sigma_i \sigma_j$.

“Ferromagnetic regime”: same spin is good.

Gibbs distribution $\propto e^{-E/kT}$.

Physical properties from Gibbs distribution.

Metropolis Algorithm:
At x, generate y with a single random flip.
Go to y with probability $\min(1, w(y)/w(x))$

Random walk in sample space graph (BIG GRAPH ALERT)
(not random walk in 2d grid of particles.)

Markov Chain on statespace of system.

Sampling structures and the BIG GRAPH

Sampling Algorithms $\equiv$ Random walk on BIG GRAPH. Small degree.

Vertices	Neighbors	Degree (ish)
Grid points in convex body.	Change one dimension	$2d$
Spanning Trees.	Change two edges.	$\leq \lvert V\rvert^2$ neighbors per node
Spin States.	Change one spin	$O(n)$ neighbors.

Analyzing random walks on graph.

Start at vertex, go to random neighbor.
For d-regular graph: eventually uniform.
if not bipartite. Odd /even step!

How to analyse?

Random Walk Matrix: M.
M - normalized adjacency matrix.
Symmetric, $\sum_j M[i,j] = 1$.
$M[i,j]$- probability of going to j from i.

Probability distribution at time t: v_t.
$v_{t+1} = Mv_t$ Each node is average over neighbors.

Evolution? Random walk starts at 1, distribution $e_1 = [1,0,\ldots,0]$.
$M^t v_1 = \frac{1}{N} v_1 + \sum_{i>1} \lambda_i^t \alpha_i v_i$.
$v_1 = [\frac{1}{N},\ldots,\frac{1}{N}]$ $\rightarrow$Uniform distribution.

Doh! What if bipartite?
Negative eigenvalues of value -1: $(+1,-1)$ on two sides.
Side question: Why the same size? Assumed regular graph.

Fix-it-up chappie!

"Lazy" random walk: With probability $1/2$ stay at current vertex.

Evolution Matrix: $\frac{I+M}{2}$
Eigenvalues: $\frac{1+\lambda_i}{2}$
$\frac{1}{2}(I+M)v_i = \frac{1}{2}(v_i + \lambda_i v_i) = \frac{1+\lambda_i}{2} v_i$
Eigenvalues in interval $[0,1]$.

Spectral gap: $\frac{1-\lambda_2}{2} = \frac{\mu}{2}$.

Uniform distribution: $\pi = [\frac{1}{N},\ldots,\frac{1}{N}]$
Distance to uniform: $d_1(v_t,\pi) = \sum_i |(v_t)_i - \pi_i|$

"Rapidly mixing": $d_1(v_t,\pi) \leq \varepsilon$ in **poly**$(\log N, \log \frac{1}{\varepsilon})$ time.
When is chain rapidly mixing?

Another measure: $d_2(v_t,\pi) = \sum_i ((v_t)_i - \pi_i)^2$.
Note: $d_1(v_t,\pi) \leq \sqrt{N} d_2(v_t,\pi)$
n – "size" of vertex, $\mu \geq \frac{1}{p(n)}$ for poly $p(n)$, $t = O(p(n)\log N)$.

$$d_2(v_t,\pi) = |A^t e_1 - \pi|^2 \leq \left(\frac{(1+\lambda_2)}{2}\right)^{2t} \leq (1 - \frac{1}{2p(n)})^{2t} \leq \frac{1}{\text{poly}(N)}$$

Rapidly mixing with big ($\geq \frac{1}{p(n)}$) spectral gap.

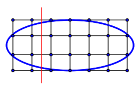
Rapid mixing, volume, and surface area..

Recall volume of convex body.
Grid graph on grid points inside convex body.

Recall Cheeger: $\frac{\mu}{2} \leq h(G) \leq \sqrt{2\mu}$.

Lower bound expansion $\rightarrow$ lower bounds on spectral gap μ
$\rightarrow$ Upper bound mixing time.

$h(G) \approx \frac{\text{Surface Area}}{\text{Volume}}$



Isoperimetric inequality.

$\text{Vol}_{n-1}(S,\overline{S}) \geq \frac{\min(\text{Vol}(S),\text{Vol}(\overline{S}))}{diam(P)}$

Edges $\propto$ surface area, Assume $Diam(P) \leq p'(n)$
$\rightarrow h(G) \geq 1/p'(n)$
$\rightarrow \mu > 1/2p'(n)^2$
$\rightarrow O(p'(n)^2 \log N)$ convergence for Markov chain on BIG GRAPH.
$\rightarrow$ Rapidly mixing chain:

Khachiyan's algorithm for counting partial orders.

Given partial order on $x_1,\ldots,x_n$.

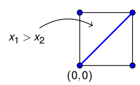
Sample from uniform distribution over total orders.

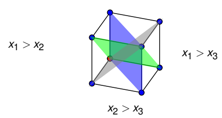
Start at an ordering.
Swap random pair and go if consistent with partial order.

Rapidly mixing chain?

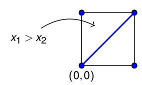
Map into d-dimensional unit cube.

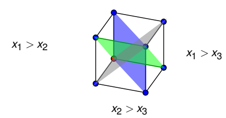
$x_i < x_j$ corresponds to halfspace (one side of hyperplane) of cube.
"dimension i = dimension j"
total order is intersection of n halfspaces.
each of volume: $\frac{1}{n!}$.
since each total order is disjoint
and together cover cube.





Poly time.





Each order takes $\frac{1}{n!}$ volume.

Number of orders $\equiv$ volume of intersection of partial order relations.

Diameter: $O(\sqrt{n})$

Isoperimetry:
$Vol_n(S) = \frac{|S|}{n!}$. and $Vol_{n-1}(S,\overline{S}) \geq \frac{Vol_n S}{diam(P)}$.
$Vol_{n-1}(S,\overline{S}) = \frac{E(S,\overline{S})}{(n-1)!} \geq \frac{|S|}{n!\sqrt{n}}$
Edge Expansion: the degree d is $O(n^2)$,
$h(S) = \frac{|E(S,\overline{S})|}{d|S|} \geq \frac{1}{n^{7/2}}$ Mixes in time $O(n^7 \log N)$ = $O(n^8 \log n)$.
Do the polynomial dance!!!

Miller-Rabin.

Pick a random a, check if $a^{N-1} = 1 \pmod{N}$.

If N not prime and any a fails test, half the a's fail test.

Repeat k times.

n possibilities, $\log n$ bits, half the possibilities are good.

Total: $O(k \log n)$ random bits. Failure probability is $1/2^k$.

Another view: n-vertex degree d graph with $\lambda_1 - \lambda_2 \geq \Omega(\sqrt{d})$.

Half the vertices correspond to good a's.

Choose random vertex, and do random walk of length ck.
$\ell = \log n + ck$ random bits.

Given a random walk of length ℓ in an expander graph (N, d, λ), what is the probability that you stay in a bad set B, with $|B|/n \leq \beta = \frac{1}{2}$, for all the steps?

$Pr[$ stay in $B] \leq ((1-\lambda)\sqrt{\beta} + \lambda)^{\ell}$

Proof: set up walk.

Claim: $Pr[$ stay in $B] \leq ((1-\lambda)\sqrt{\beta} + \lambda)^{\ell}$

B_i - event in B at step i.

$\hat{B}$ is diagonal matrix with 1's corresponding to $i \in B$.

Consider random walk that truncates when it hits $v \notin B$.

Distribution over B at beginning: $\hat{B}\mathbf{1}$. $\mathbf{1} = (\mathbf{1}/\mathbf{N}, \ldots, \mathbf{1}/\mathbf{N})$
$1/N$ for each vertex in B.

Distribution over B at time 2, $\hat{B}A\hat{B}\mathbf{1}$

At time ℓ, $(\hat{B}A)^{\ell}\hat{B}\mathbf{1}$.

Total probability in B: $\|(\hat{B}A)^{\ell}\hat{B}\mathbf{1}\|_{\mathbf{1}}$

Will prove: $\|(\hat{B}A)^{\ell}\hat{B}\mathbf{1}\|_{\mathbf{2}} \leq \frac{((\mathbf{1}-\lambda)\sqrt{\beta}+\lambda)^{\ell}\sqrt{\beta}}{\sqrt{\mathbf{N}}}$

plus $|x|_1 \leq \sqrt{N}|x|_2 \implies$ Claim.

Bounding the 2-norm of A.

Def: $\|B\|_2$, is $max \frac{\|Bx\|_2}{\|x\|_2}$. $\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$. $\|AB\|_2 = \|A\|_2\|B\|_2$.

J scaled adjacency matrix of clique: $J_{i,j} = 1/n$.

Claim: If A is scaled adjacency matrix for λ expander.

$A = (1-\lambda)J + \lambda C$.

where $\|Cv\|_2 \leq \|v\|_2$ for all v.

Proof:
$C = \frac{1}{\lambda}(A - (1-\lambda)J)$.
Consider $v = u + w$, with $u = \alpha 1$ and $w \perp 1$.
$Cu = \frac{1}{\lambda}(A - (1-\lambda)J)u = \frac{1}{\lambda}(1 - (1-\lambda))u = u$
$w' = Aw$, $|w'|_2^2 = w^T AAw = \lambda^2 |w|_2^2$.
$\implies \|Cw\|_2 = \frac{1}{\lambda}\|Aw\|_2 \leq \|w\|_2$. □
Remember: $|B|/n = \beta$.

$\hat{B}A = \hat{B}((1-\lambda)J + \lambda C)$ and $\|\hat{B}J\|_2 \leq \sqrt{\beta}$ and $\|\hat{B}C\| \leq 1$.
$\implies \|BA\|_2 \leq (1-\lambda)\sqrt{\beta} + \lambda$

Also, $\|\hat{B}\mathbf{1}\|_{\mathbf{2}} \leq \frac{\sqrt{\beta}}{\sqrt{\mathbf{n}}}$. $\implies \|(\hat{B}A)^{\ell}\hat{B}\mathbf{1}\|_{\mathbf{2}} \leq \frac{((\mathbf{1}-\lambda)\sqrt{\beta}+\lambda)^{\ell}\sqrt{\beta}}{\sqrt{\mathbf{N}}}$

Also.

Flip k coins: don't get heads with probability $(1/2)^k$.

Analagous statement for expanders: $(f(\lambda, 1/2))^k$, where $f(\lambda, 1/2)) < 1$.

Flip k coins: get roughly $k/2$ heads.

Something analagous for walk in expanders.

Summary.

Eigenvectors for hypercubes.
Tight example for LHI of Cheeger. Eigenvectors for cycle.
Tight example for RHI of Cheeger.

Random Walks and Sampling.

Eigenvectors, Isoperimetry of Volume, Mixing.

Partial Order Application.