Today

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Balls in Bins.

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Power of two choices.

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Cuckoo hashing.

Chernoff implications.

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Note: counting gives better bound.

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$k! \geq (\frac{k}{e})^k$ Roughly: $\ln k! = \sum_i \ln k \approx \int \ln k = k \ln k - k$

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Lemma: Max load is $\Theta(\log n)$ with probability $\geq 1 - \frac{1}{n}$.

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Better than Chernoff bound.

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Better than Chernoff bound. Standard deviation is as big as mean....

Power of two..

n balls in n bins.

Power of two..

n balls in n bins.

Choose two bins, pick least loaded.

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Is max load lower?

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Exponentially better!

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Exponentially better! Old bound is exponential of new bound.

Analysis.

$n/8$ balls in n bins.

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Each ball chooses two bins at random.

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View as graph.

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 Bin is vertex.

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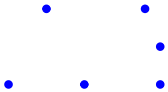

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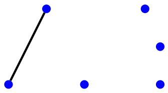

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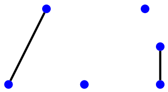

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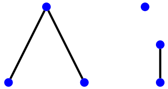

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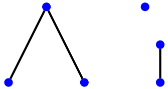

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Add edge, add one to lower endpoint's "count."

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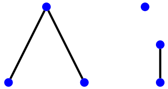

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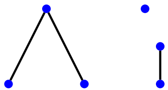

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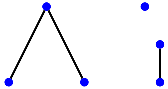

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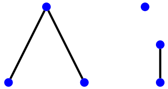

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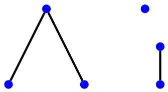

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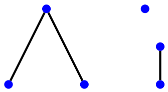

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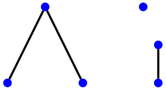

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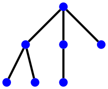

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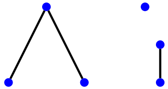

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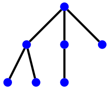

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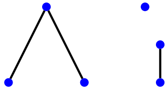

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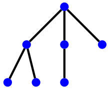

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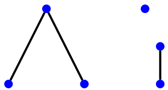

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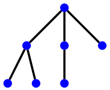

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Average induced degree is small. (E.g.: cycle degree 2)

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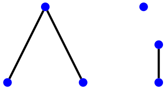

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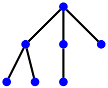

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Extend tree intuition.

Connected Component.

Claim: Component size in n vertex, $\frac{n}{8}$ edge random graph is $O(\log n)$ w/ prob. $\geq 1 - \frac{1}{n^c}$.

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Proof: Size k component, C, contains $\geq k-1$ edges.

$$\Pr[|C| \geq k] \leq \binom{n}{k} \binom{n/8}{k-1} \left(\frac{k}{n}\right)^{2(k-1)}$$

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Possible C.

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Possible C. Which edges. Prob. both endpoints inside C.

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Proof: Size k component, C, contains $\geq k-1$ edges.

$$\Pr[|C| \geq k] \leq \binom{n}{k}\binom{n/8}{k-1}\left(\frac{k}{n}\right)^{2(k-1)}$$

Possible C. Which edges. Prob. both endpoints inside C.

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Choose $k = -(c+1)\log_{.93} n$ make probability $\leq 1/n^c$.

Not dense.

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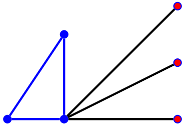

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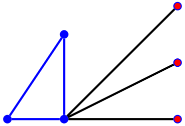

Induced degree of node in blue subset is 2,

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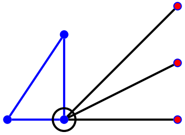

Induced degree of node in blue subset is 2, not 5!

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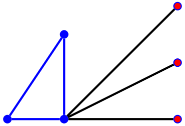

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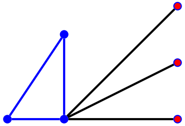

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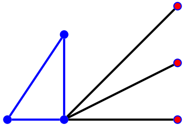

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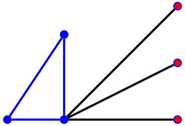

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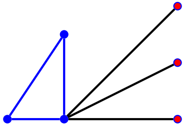

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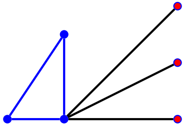

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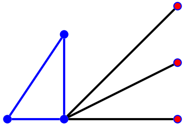

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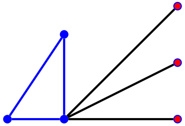

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Removal Process!

Random Graph: Component size is $c \log n$ and max-induced degree is 8 w.h.p.

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Claim: $O(\log X)$ iterations where X is max component size.

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Average induced degree 8

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Height of ball, h_i, is load of bin when it is placed in bin.

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Induction:

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Induction: Previous removed edges(ball) induce load $\leq 16(r_i - 1)$.

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Power of two choices.

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X is $O(\log n)$ with high probability.

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Max load is $O(\log \log n)$.

Cuckoo hashing.

Hashing with two choices: max load $O(\log \log n)$.

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Cuckoo hashing:
Array.

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Cuckoo hashing:
Array. Two hash functions h_1, h_2.

Cuckoo hashing.

Hashing with two choices: max load $O(\log\log n)$.

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Insert x: place in $h_1(x)$ or $h_2(x)$ if space.

Cuckoo hashing.

Hashing with two choices: max load $O(\log\log n)$.

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If go too long. Fail.

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If go too long. Fail. Rehash entire hash table.

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C_ℓ - event of cycle of length ℓ.

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See you on Thursday...