Today

Load balancing.

Balls in Bins.

Power of two choices.

Cuckoo hashing.

Chernoff implications.

For Bernoulli variables $X_1,\ldots,X_n$ with $E[X_i]=p_i$, $X=\sum_i X_i$, $\mu=\sum_i p_i$,
If $0<\delta<1$,

$$Pr[X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2\mu}{3}}.$$

If $R>6\mu$,

$$Pr[X \geq R] \leq \left(\frac{1}{2}\right)^R.$$

Interesting parameters:

If $\mu=1$, then $Pr[X \geq \log 1/p] \leq p$ (if $\log(1/p)>6$).
Example: $p_i=1/n$, then $Pr[X \geq \log N] \leq \frac{1}{N}$.

Note: counting gives better bound.

Large mean.

For Bernoulli variables $X_1,\ldots,X_n$ with $E[X_i]=p_i$, $X=\sum_i X_i$, $\mu=\sum_i p_i$,
If $0<\delta<1$,

$$Pr[X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2\mu}{3}}.$$

Large μ in deviations: Bernoulli $X_1+\cdots X_n$, with $p=1/2$.

Bound with $\delta=t/\sqrt{n}$, implies $Pr[(X-n/2) \geq t\sqrt{n}/2] \leq e^{-\frac{t^2}{6}}$.
With $t=\Theta(\sqrt{\log 1/p}) \implies Pr[(X-n/2) \geq \sqrt{tn/2}] \leq p$.
Failure probability of $1/N$, need $t=\Theta(\sqrt{\log N})$.

Different analysis gives different constants: e.g., subgaussian distributions give Hoeffdings.

How many samples? Bernoulli $X_1+\cdots X_n$, with $p=1/2$.

Bound with $\delta=\varepsilon$, implies $Pr[(X-n/2) \geq (\varepsilon)n/2] \leq e^{-\frac{\varepsilon^2 n}{6}}$.
$n=\Theta(\frac{\sqrt{\log 1/p}}{\varepsilon^2}) \implies Pr[(X-n/2) \geq \sqrt{tn/2}] \leq p$.
Failure probability of $1/N$, need $n \geq \Theta(\frac{\sqrt{\log N}}{\varepsilon^2})$.

Some bounds.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k$$

$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdot 1} = \frac{n}{k}\cdot\frac{n-1}{k-1}\cdots\frac{n-k+1}{1} \geq \frac{n}{k}\cdot\frac{n}{k}\cdots\frac{n}{k}$

$n(n-1)\cdots(n-k+1) \leq n^k$

$k! \geq (\frac{k}{e})^k$ Roughly: $\ln k! = \sum_i \ln k \approx \int \ln k = k\ln k - k$

Simplest..

Load balance: m balls in n bins.

For simplicity: n balls in n bins.

Round robin: load 1 !

Centralized! Not so good.

Uniformly at random? Average load 1.

Max load?

n. Uh Oh!

Max load with probability $\geq 1-\delta$?

$\delta=\frac{1}{n^c}$ for today. c is 1 or 2.

Balls in bins.

For each of n balls, choose random bin: X_i balls in bin i.

$Pr[X_i \geq k] \leq \sum_{S\subseteq[n],|S|=k} Pr$[balls in S chooses bin i]

From Union Bound: $Pr[\cup_i A_i] \leq \sum_i Pr[A_i]$

Pr[balls in S chooses bin i] $=(\frac{1}{n})^k$ and $\binom{n}{k}$ subsets S.

$$\begin{aligned}\Pr[X_i \geq k] &\leq \binom{n}{k}\left(\frac{1}{n}\right)^k \\ &\leq \frac{n^k}{k!}\left(\frac{1}{n}\right)^k = \frac{1}{k!}\end{aligned}$$

Choose k, so that $Pr[X_i \geq k] \leq \frac{1}{n^2}$.
$Pr[\text{any } X_i \geq k] \leq n \times \frac{1}{n^2} = \frac{1}{n} \rightarrow$ max load $\leq k$ w.p. $\geq 1-\frac{1}{n}$

Solving for k

$Pr[X_i \geq k] \leq \frac{1}{k!} \leq 1/n^2$?

What is upper bound on max-load k?

Lemma: Max load is $\Theta(\log n)$ with probability $\geq 1 - \frac{1}{n}$.
$k! \geq n^2$ for $k = 2e\log n$
(Recall $k! \geq (\frac{k}{e})^k$.)
$\implies \frac{1}{k!} \leq (\frac{e}{k})^k \leq \left(\frac{1}{2\log n}\right)^k$
If $\log n \geq 1$, then $k = 2e\log n$ suffices.

Also: $k = \Theta(\log n / \log\log n)$ suffices as well.

$k^k \to n^c$.

Actually Max load is $\Theta(\log n / \log\log n)$ w.h.p.

(W.h.p. - means with probability at least $1 - O(1/n^c)$ for today.)

Better than Chernoff bound. Standard deviation is as big as mean....

Power of two..

n balls in n bins.

Choose two bins, pick least loaded.

still distributed, but a bit less than not looking.

Is max load lower? Yes? No? Yes.

How much lower?

$\log n/2$? $\sqrt{\log n}$? $O(\log\log n)$?

$O(\log\log n)$! ! ! !

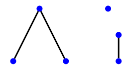
Exponentially better! Old bound is exponential of new bound.

Analysis.

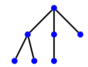
$n/8$ balls in n bins.

Each ball chooses two bins at random.
picks least loaded.

View as graph.
Bin is vertex.
Each ball is edge.
Analysis Intuition:
Add edge, add one to lower endpoint's "count."



Max load is max vertices count.
If max count is k.
neighbors with counts $\geq k-1, k-2, k-3, \ldots$.
and so on!
No cycles and max-load $k \to \geq 2^{k/2}$ nodes in tree.
No connected component of size X and no cycles
$\implies$ max load $O(\log X)$.



Will show:
Max conn. comp is $O(\log n)$ w.h.p.
Average induced degree is small. (E.g.: cycle degree 2)
Extend tree intuition.

Connected Component.

Claim: Component size in n vertex, $\frac{n}{8}$ edge random graph is $O(\log n)$ w/ prob. $\geq 1 - \frac{1}{n^c}$.
Proof: Size k component, C, contains $\geq k-1$ edges.

$$\Pr[|C| \geq k] \leq \binom{n}{k}\binom{n/8}{k-1}\left(\frac{k}{n}\right)^{2(k-1)}$$

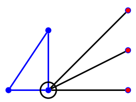
Possible C. Which edges. Prob. both endpoints inside C.

$$\begin{aligned}\Pr[|C| \geq k] &\leq \frac{n}{k}\binom{n}{k}\binom{n/8}{k}\left(\frac{k}{n}\right)^{2k} \\ &\leq \frac{n}{k}\left(\frac{ne}{k}\right)^k\left(\frac{ne}{8k}\right)^k\left(\frac{k}{n}\right)^{2k} = \frac{n}{k}\left(\frac{e^2}{8}\right)^k \leq \frac{n}{k}(0.93)^k\end{aligned}$$

Choose $k = -(c+1)\log_{.93} n$ make probability $\leq 1/n^c$.

Not dense.

Induced degree of node on subset, S, is degree of internal edges.



Induced degree of node in blue subset is 2, not 5!

Claim: Average induced degree on any subset of nodes is ≤ 8 with probability $\geq 1 - O(\frac{1}{n^2})$.

Proof: Induced degree ≥ 8
$\to 4k$ internal edges for subset of size k.

$$\Pr[\text{dense } S] \leq \binom{n}{k}\binom{n/8}{4k}\left(\frac{k}{n}\right)^{8k} \leq \left(\frac{e^{1.25}}{32}\right)^{4k}\left(\frac{k}{n}\right)^{3k} \leq \left(\frac{k}{n}\right)^{3k}$$

Starts at $1/n^3$, decreasing till $k \leq n/8$ (at least)
$\to$ Total $O(1/n^2)$.

Removal Process!

Random Graph: Component size is $c\log n$ and max-induced degree is 8 w.h.p.

Process: Remove degree ≤ 16 nodes
and incident edges. Repeat.
Claim: $O(\log X)$ iterations where X is max component size.

For any connected component:
Average induced degree 8 $\to$ half nodes w/degree ≤ 16.
$\to$ half nodes removed in each iteration.
$\to \log X$ iterations to remove all nodes.

Claim: Max load is $O(\log\log n)$ w.h.p.

Recall edge corresponds to ball.
Height of ball, h_i, is load of bin when it is placed in bin.
Corresponding edge removed in iteration r_i.
Property: $h_i \leq 16r_i$.
Case $r_i = 1$ - only 16 balls incident to bin $\to h_i \leq 16$.
Induction: Previous removed edges(ball) induce load $\leq 16(r_i - 1)$.
$+16$ edges/balls this iteration.
$\to h_i \leq 16r_i$.

Power of two choices.

Max load: $\log X$ where X is max component size.

X is $O(\log n)$ with high probability.

Max load is $O(\log\log n)$.

Cuckoo hashing.

Hashing with two choices: max load $O(\log\log n)$.

Cuckoo hashing:
Array. Two hash functions h_1, h_2.

Insert x: place in $h_1(x)$ or $h_2(x)$ if space.
Else bump elt y in $h_i(x)$ u.a.r. for $i \in [1,2]$.
Place x in $h_i(x)$. Bump y: place y in $h_j(y)$ where $j \neq i$ if space.
Else bump y' in $h_j(y)$.

If go too long. Fail. Rehash entire hash table.
Fails if cycle for insert.

C_ℓ - event of cycle of length ℓ.

Use: $m \leq n/8$ items.

$$\Pr[C_\ell] \leq \binom{m}{\ell}\binom{n}{\ell}\left(\frac{\ell}{n}\right)^{2(\ell)} \leq \left(\frac{e^2}{8}\right)^{\ell} \quad (1)$$

Rehash every $\Omega(n)$ inserts (if $\leq n/8$ items in table.)
$O(1)$ time on average.

Power of Two: Large Load!

What if m - number of balls is much larger than n?

Previous analysis $O(m \log\log n/n)$ (versus round robin m/n.)

Can do $O(m/n) + O(\log\log n)$.

Large Load Balls in Bins: $m/n + O(\sqrt{(m/n)\log n})$.

Imbalance becomes small order term as average gets large.
Large means m/n and relatively small variance $\sqrt{m/n}$.

Two choices: m/n + ???

$m/n + O(\log\log n)$!

Sum up

Balls in bins: $\Theta(\log n/\log\log n)$ load.

Power of two: $\Theta(\log\log n)$.

Cuckoo hashing.

See you on Thursday...