Linear Program.

How?

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How? From lecture warmup.

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Complementary slackness conditions imply optimality.

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Is the path fundamental?

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Are things as easy or as hard as $0, 1, 2, \ldots$...?

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Duality.

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Geometric View, Linear Equation, and Combinatorial.

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Geometric View, Linear Equation, and Combinatorial.

Today: Strong Duality from Geometry.

Simplex Algorithm

$$\max c \cdot x.$$

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$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

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Start at feasible point where n equations are satisfied.

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Another view: intersection of n hyperplanes.

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Drop one equation:

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Drop one equation:
Points on line satisfy $n-1$ ind. equations.

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Drop one equation:
Points on line satisfy $n-1$ ind. equations.
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Move in direction that increases objective.

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Start at feasible point where n equations are satisfied.

E.g., $x = 0$.
This is a point.
Another view: intersection of n hyperplanes.

Drop one equation:
Points on line satisfy $n-1$ ind. equations.
Intersection of $n-1$ hyperplanes.

Move in direction that increases objective.

Until new tight constraint.

Simplex Algorithm

$$\max c \cdot x.$$

$$Ax \leq b$$
$$x \geq 0$$

Start at feasible point where n equations are satisfied.

E.g., $x = 0$.
This is a point.
Another view: intersection of n hyperplanes.

Drop one equation:
Points on line satisfy $n-1$ ind. equations.
Intersection of $n-1$ hyperplanes.

Move in direction that increases objective.

Until new tight constraint.

No direction increases objective.

Hyperplane View

$x + y + z \leq 1$

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane?

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1, 1, 1)$.

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Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1, 1, 1)$.

Why?

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1, 1, 1)$.

Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1,1,1)$.

Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
(a,b,c) where $a + b + c = 1$.

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1,1,1)$.

Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
(a, b, c) where $a + b + c = 1$.
(a', b', c') where $a' + b' + c' = 1$.

Hyperplane View

$x+y+z \leq 1$

On one side of hyperplane defined by $x+y+z=1$.

Normal to hyperplane? $(1,1,1)$.

Why? Normal: $u \cdot (v-w) = 0$ for any v, w in hyperplane.
(a,b,c) where $a+b+c=1$.
(a',b',c') where $a'+b'+c'=1$.
$(a'-a, b'-b, c'-c) \cdot (1,1,1) = (a'+b'+c'-(a+b+c)) = 0$.

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1, 1, 1)$.

Why?   Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
  (a, b, c) where $a + b + c = 1$.
  (a', b', c') where $a' + b' + c' = 1$.
  $(a' - a, b' - b, c' - c) \cdot (1, 1, 1) = (a' + b' + c' - (a + b + c)) = 0$.

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1, 1, 1)$.

Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
(a, b, c) where $a + b + c = 1$.
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$(a' - a, b' - b, c' - c) \cdot (1, 1, 1) = (a' + b' + c' - (a + b + c)) = 0$.

Normal to $mx + ny + pz = C$?

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

Normal to hyperplane? $(1, 1, 1)$.

Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
(a, b, c) where $a + b + c = 1$.
(a', b', c') where $a' + b' + c' = 1$.
$(a' - a, b' - b, c' - c) \cdot (1, 1, 1) = (a' + b' + c' - (a + b + c)) = 0$.

Normal to $mx + ny + pz = C$? (m, n, p)

Hyperplane View

$x + y + z \leq 1$

On one side of hyperplane defined by $x + y + z = 1$.

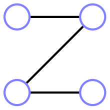
Normal to hyperplane? $(1, 1, 1)$.

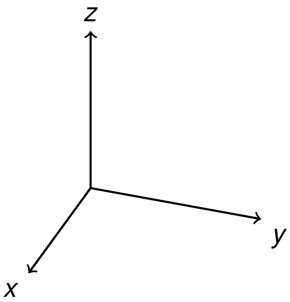
Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
(a, b, c) where $a + b + c = 1$.
(a', b', c') where $a' + b' + c' = 1$.
$(a' - a, b' - b, c' - c) \cdot (1, 1, 1) = (a' + b' + c' - (a + b + c)) = 0$.

Normal to $mx + ny + pz = C$? (m, n, p)

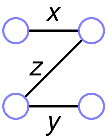
Points in hyperplane are related by nullspace of row.

Maximum matching and simplex.

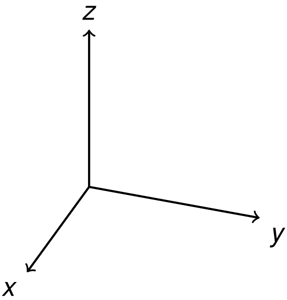




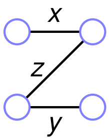
Maximum matching and simplex.



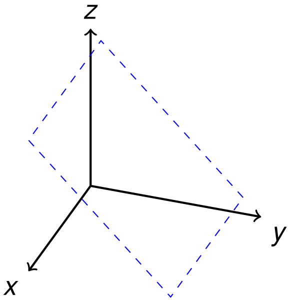
$$\max x+y+z$$
$$x \leq 1$$
$$x+z \leq 1$$
$$z+y \leq 1$$
$$y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Maximum matching and simplex.

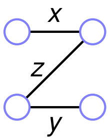


$$\max x + y + z$$
$$x \leq 1$$
$$x + z \leq 1$$
$$z + y \leq 1$$
$$y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

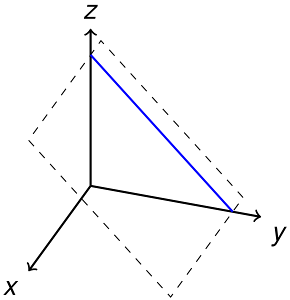


Blue constraints intersect.

Maximum matching and simplex.

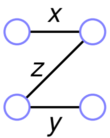


$$\max x + y + z$$
$$x \le 1$$
$$x + z \le 1$$
$$z + y \le 1$$
$$y \le 1$$
$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$



Blue constraints intersect.

Maximum matching and simplex.



$$\max x + y + z$$
$$x \leq 1$$
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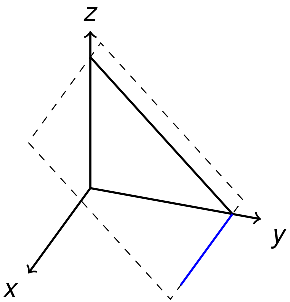

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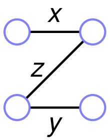
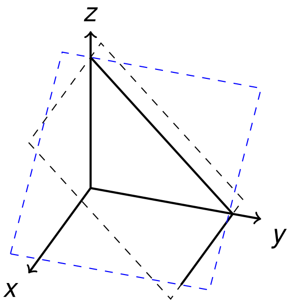
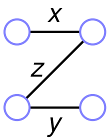


$$\max x + y + z$$
$$x \le 1$$
$$x + z \le 1$$
$$z + y \le 1$$
$$y \le 1$$
$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$

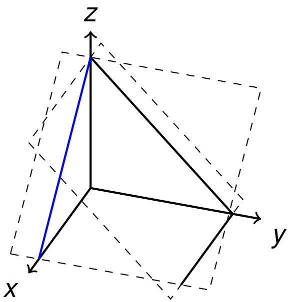


Blue constraints intersect.

Maximum matching and simplex.

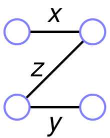


$$
\begin{aligned}
\max x + y + z \\
x \le 1 \\
x + z \le 1 \\
z + y \le 1 \\
y \le 1 \\
x \ge 0 \\
y \ge 0 \\
z \ge 0
\end{aligned}
$$

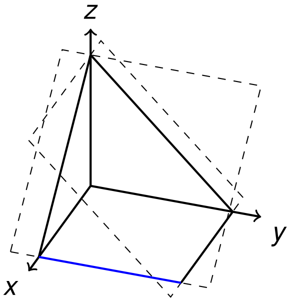


Blue constraints intersect.

Maximum matching and simplex.

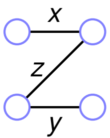


$$\max x + y + z$$
$$x \leq 1$$
$$x + z \leq 1$$
$$z + y \leq 1$$
$$y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints intersect.

Maximum matching and simplex.



$$\max x + y + z$$
$$x \leq 1$$
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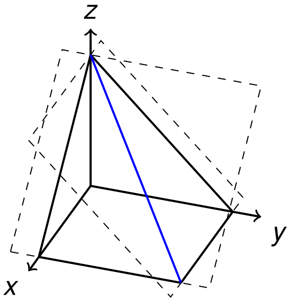
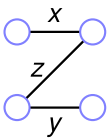


Blue constraints intersect.

Maximum matching and simplex.



$$\max x + y + z$$
$$x \leq 1$$
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$$x \geq 0$$
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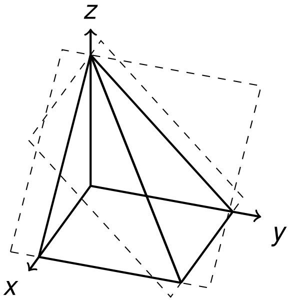
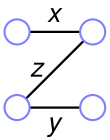
$$z \geq 0$$

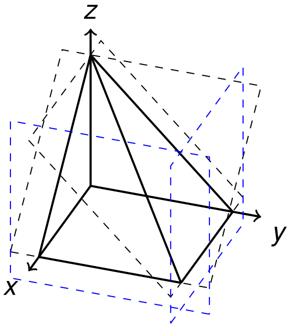


Blue constraints intersect.

Maximum matching and simplex.

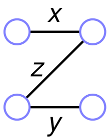


$$\max x + y + z$$
$$x \leq 1$$
$$x + z \leq 1$$
$$z + y \leq 1$$
$$y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints redundant.

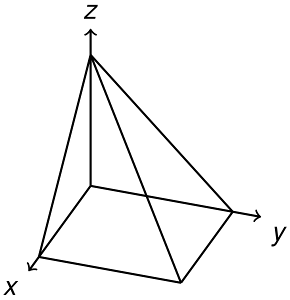
Maximum matching and simplex.



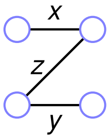
$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Maximum matching and simplex.



$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
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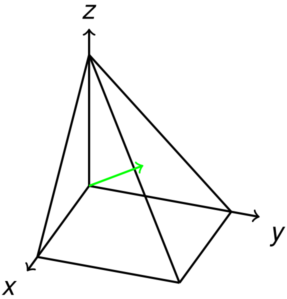
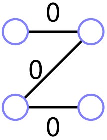


Maximum matching and simplex.

$\max x + y + z$

$x + z \leq 1$

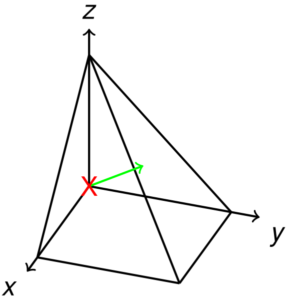
$z + y \leq 1$



$x \geq 0$

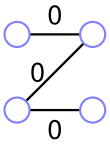
$y \geq 0$

$z \geq 0$



Blue constraints tight.

Maximum matching and simplex.



$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
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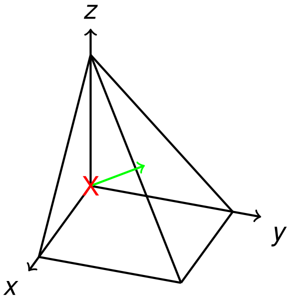
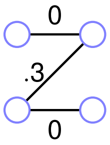


Blue constraints tight.

Maximum matching and simplex.



$$\max x + y + z$$

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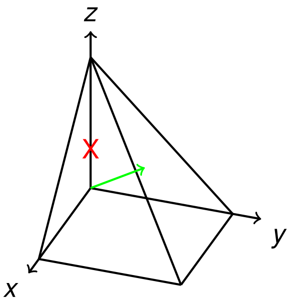

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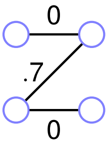
Maximum matching and simplex.



$$\max x + y + z$$

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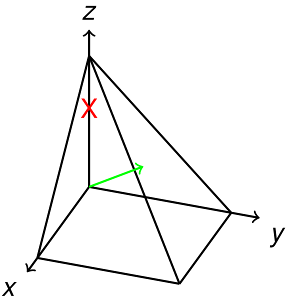

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Maximum matching and simplex.

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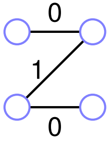

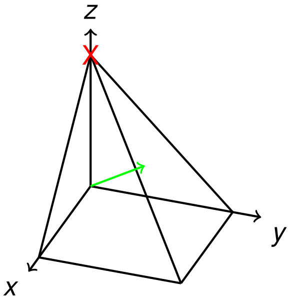

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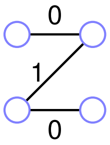
Maximum matching and simplex.



$$\max x + y + z$$

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$$x \ge 0$$
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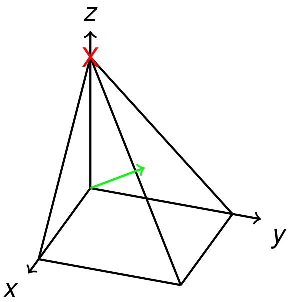
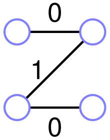


Blue constraints tight.

Maximum matching and simplex.

$$\max x + y + z$$



$$x + z \le 1$$
$$z + y \le 1$$

$$x \ge 0$$
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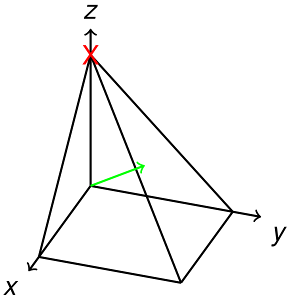

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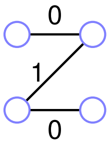

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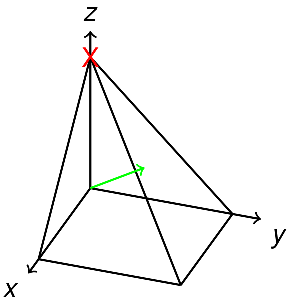

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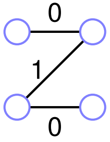

$$\max x + y + z$$

$$x + z \leq 1$$
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$$x \geq 0$$
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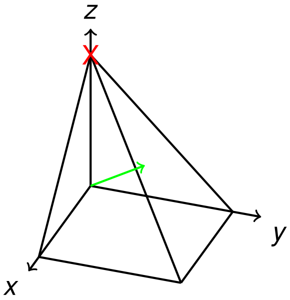

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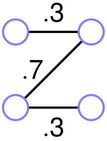
Maximum matching and simplex.

$$\max x + y + z$$



$$x + z \le 1$$
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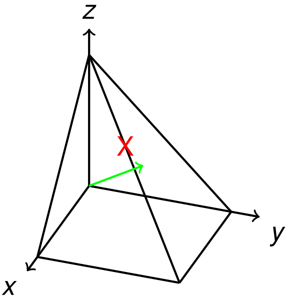

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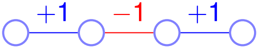
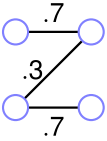


Augmenting Path.

Maximum matching and simplex.

$$\max x + y + z$$



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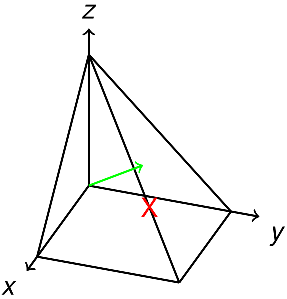

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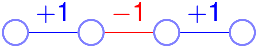

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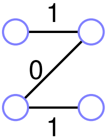

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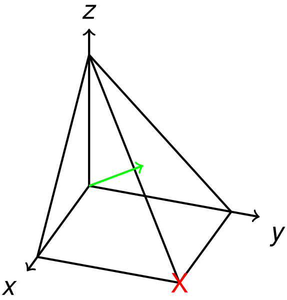

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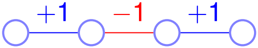

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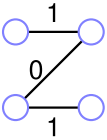

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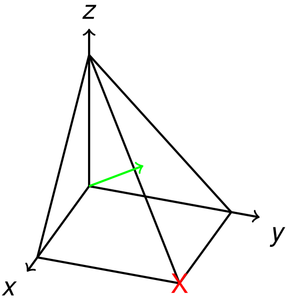

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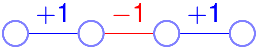

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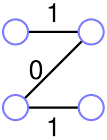

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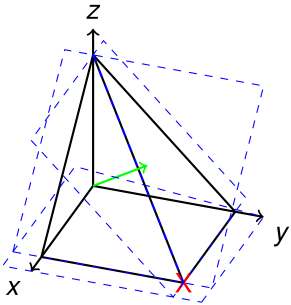

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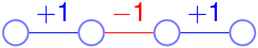

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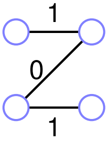

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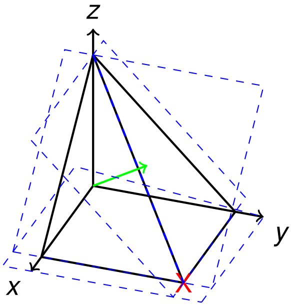

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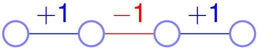

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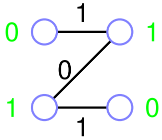

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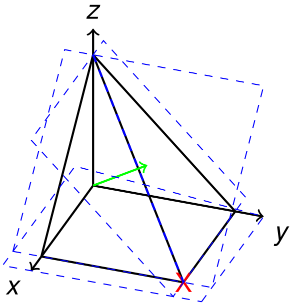

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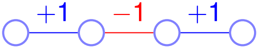
Sum: $x + z + y$.



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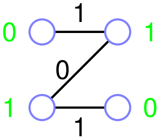

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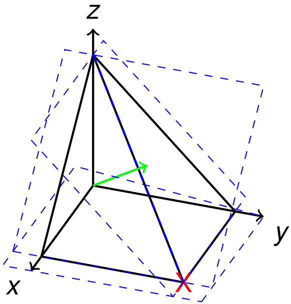

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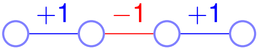

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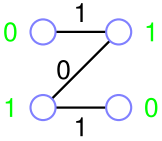

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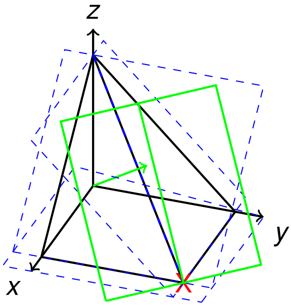

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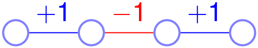

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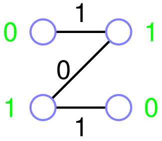

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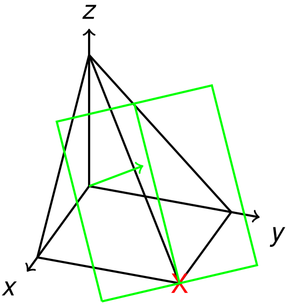

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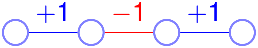

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Strong Duality

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Convex Separator.

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Farkas

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Strong Duality!!!!!

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Strong Duality!!!!! Maybe.

Linear Equations.

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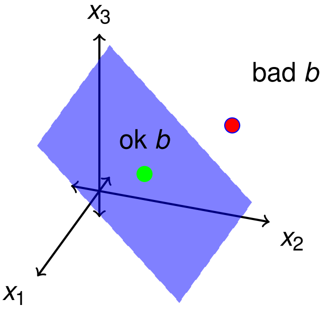

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Exercise:

$$Ax \le b, x \ge 0$$

defines a convex set of points.

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For a convex body P and a point b,
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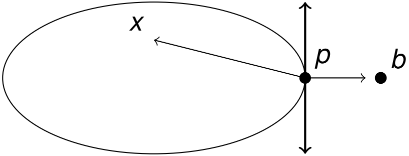

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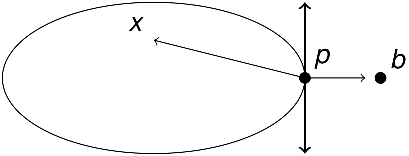

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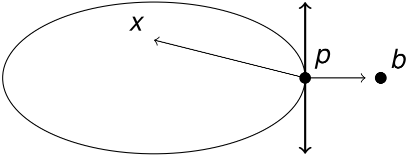

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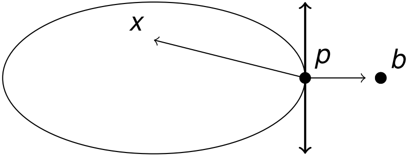

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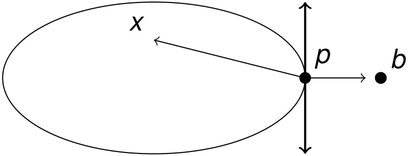

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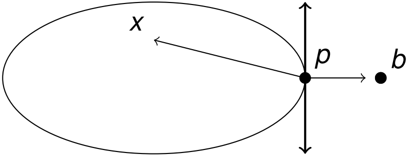

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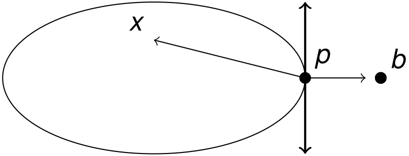

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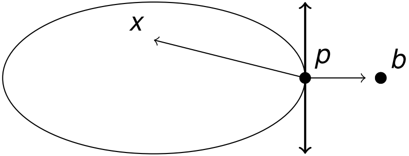

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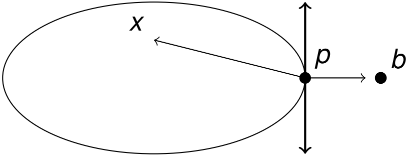

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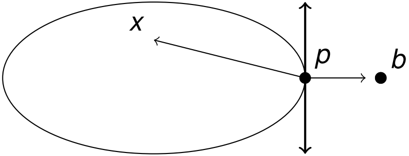

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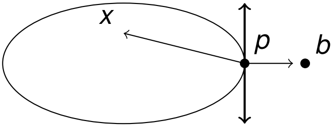

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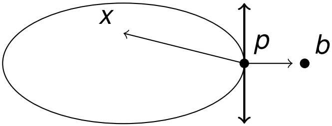

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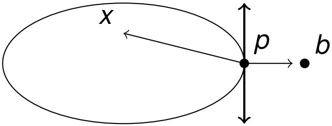

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Done or $\exists \ x \in P$ with $(x-p)^T(b-p) > 0$

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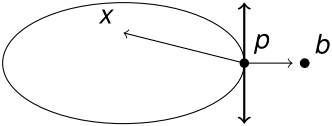
or there is point p where $(x-p)^T(b-p) \leq 0 \ \forall x \in P$.



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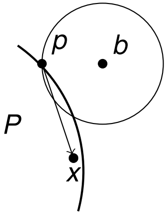
$$(x-p)^T(b-p) \geq 0$$



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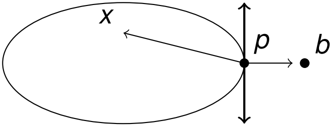

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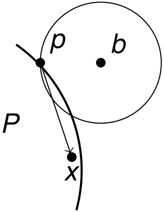

$(x-p)^T(b-p) \geq 0$
$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

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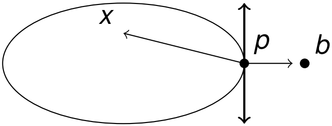

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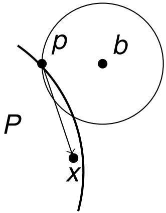

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Must be closer point b on line from p to x.

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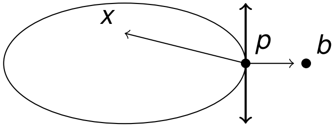

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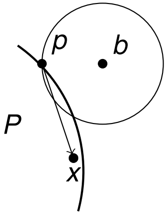

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All points on line to x are in polytope.

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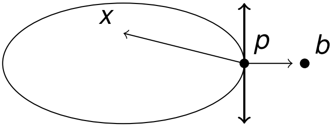

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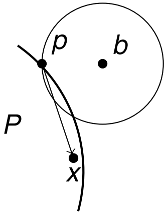

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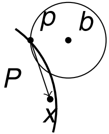
Contradicts choice of p as closest point to b in polytope.

More formally.



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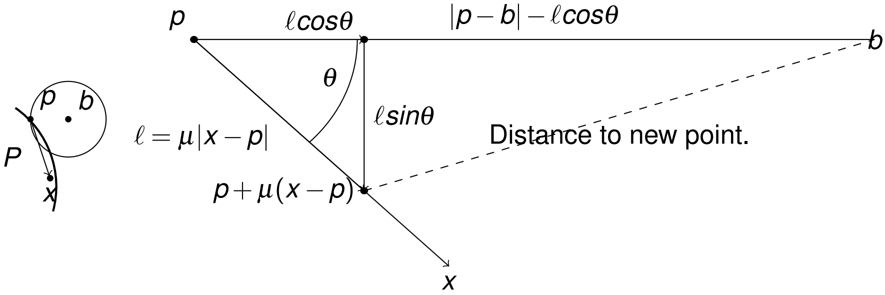

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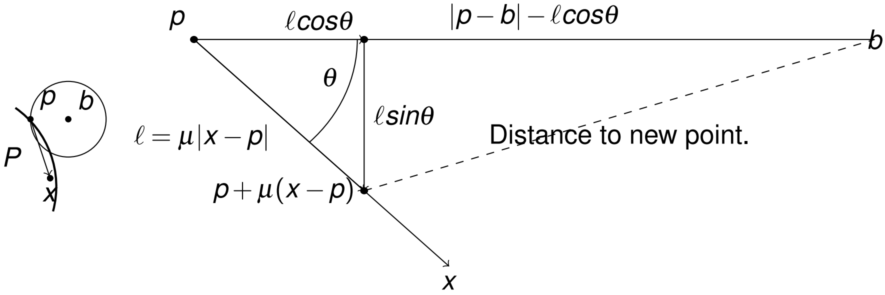
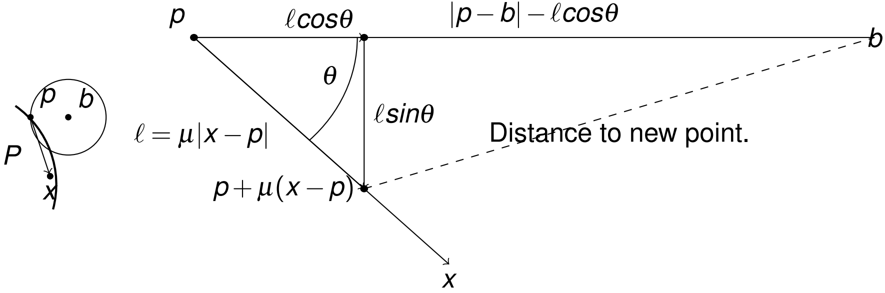


Distance to new point.

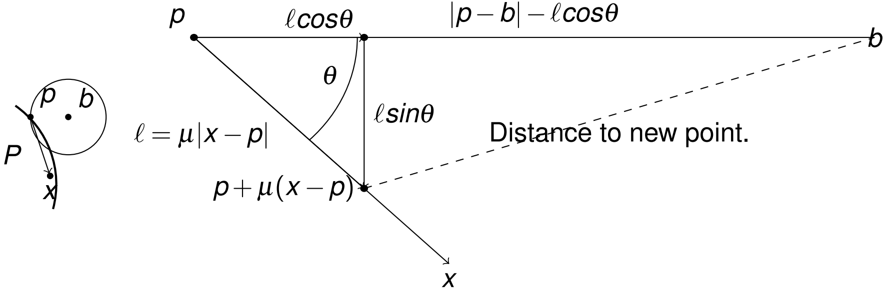
Squared distance to b from $p + (x-p)\mu$

More formally.



Squared distance to b from $p+(x-p)\mu$
point between p and x

More formally.

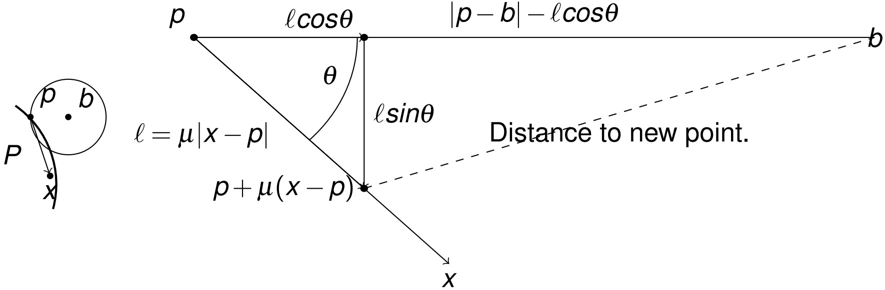


Squared distance to b from $p+(x-p)\mu$

point between p and x

$(|p-b|-\mu|x-p|cos\theta)^2+(\mu|x-p|sin\theta)^2$

More formally.



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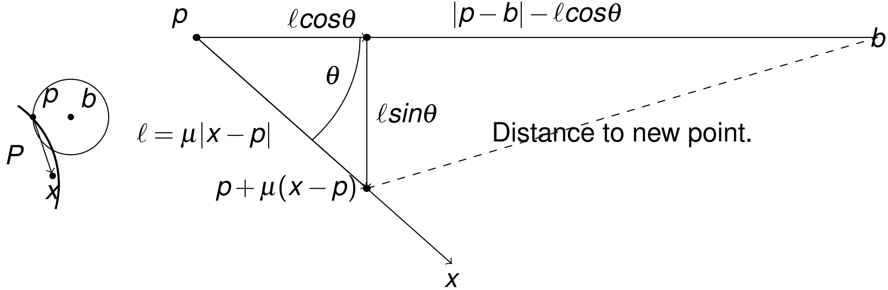

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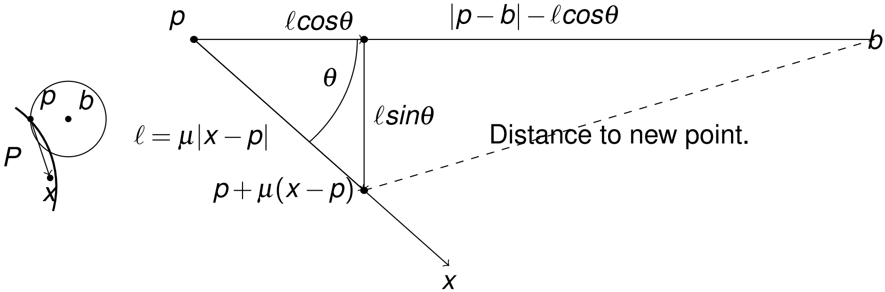
Simplify:

More formally.



Squared distance to b from $p + (x - p)\mu$

point between p and x

$(|p - b| - \mu|x - p|cos\theta)^2 + (\mu|x - p|sin\theta)^2$

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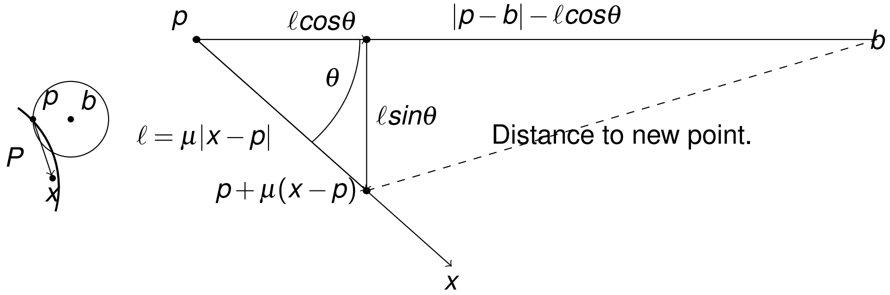
Simplify:

$|p - b|^2 - 2\mu|p - b||x - p|cos\theta + (\mu|x - p|)^2$.

More formally.



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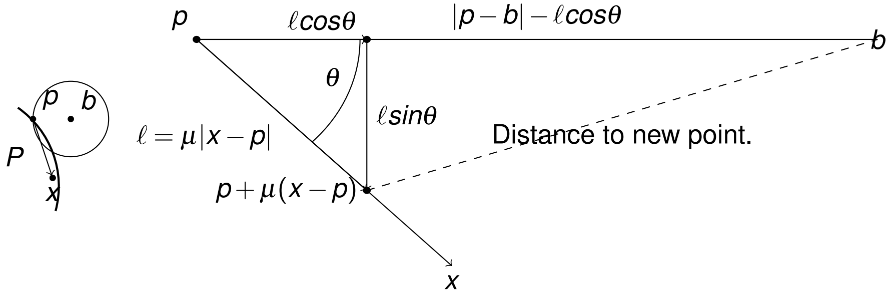
Derivative with respect to μ ...

More formally.



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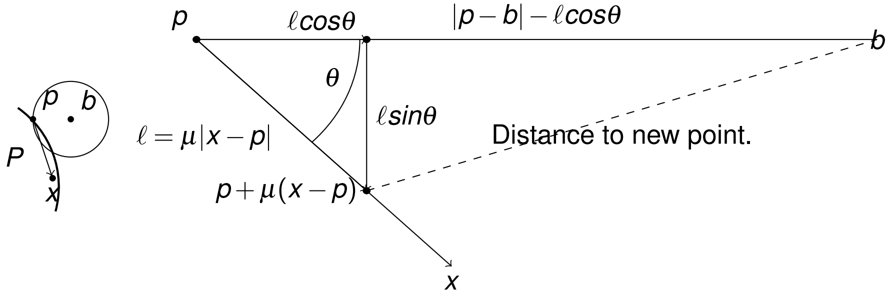
Derivative with respect to μ ...

$-2|p-b||x-p|cos\theta+2(\mu|x-p|^2).$

More formally.



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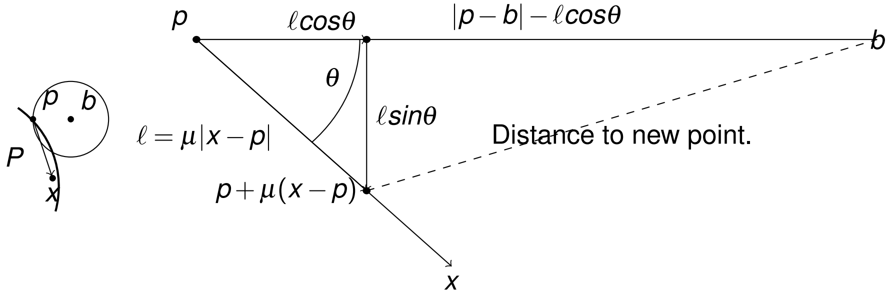

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Derivative with respect to μ ...
$-2|p-b||x-p|cos\theta+2(\mu|x-p|^2)$.
which is negative for a small enough value of μ (for positive $cos\theta$.)

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Theorems of Alternatives.

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There is a separating hyperplane between any two convex bodies.

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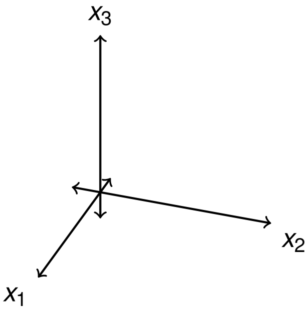
There is a separating hyperplane between any two convex bodies.

Idea: Let closest pair of points in two bodies define direction.

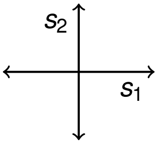
$$Ax = b,\ x \geq 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Coordinates $s = b - Ax$.

$x \geq 0$ where $s = 0$?



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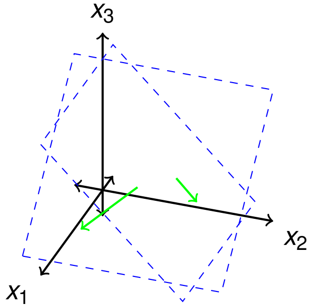

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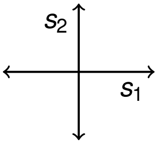

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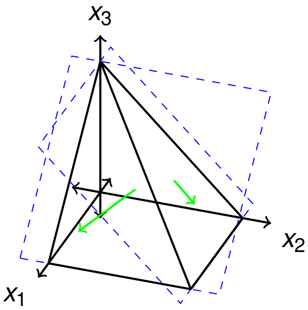

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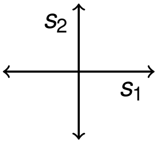

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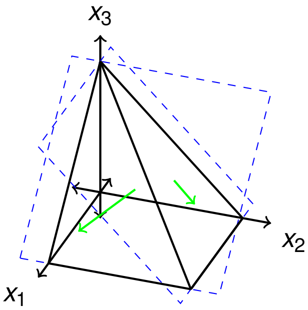

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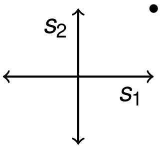

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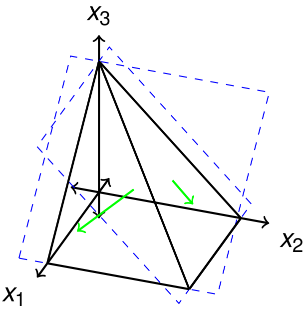

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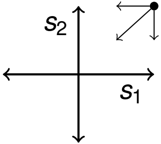

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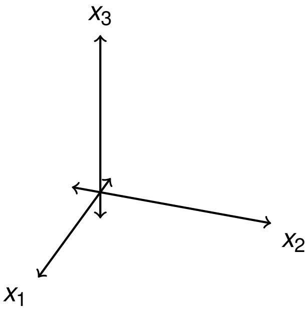

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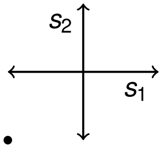

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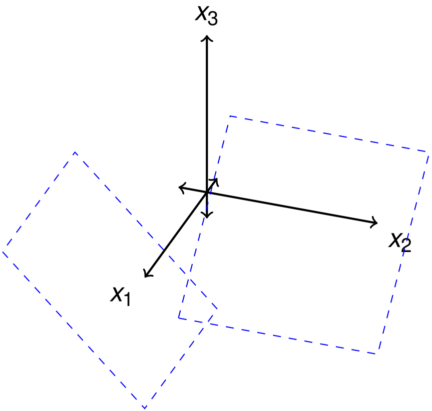

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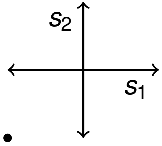

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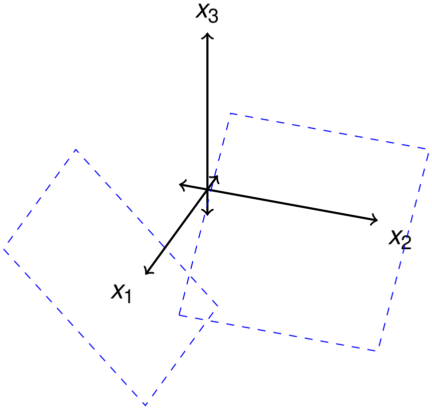

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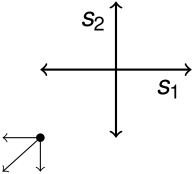

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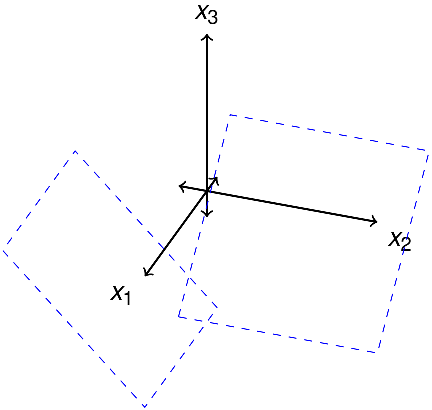

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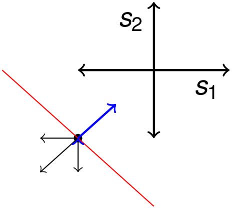

y where $y^T(b - Ax) < y^T(0) = 0$ for all $x \geq 0$

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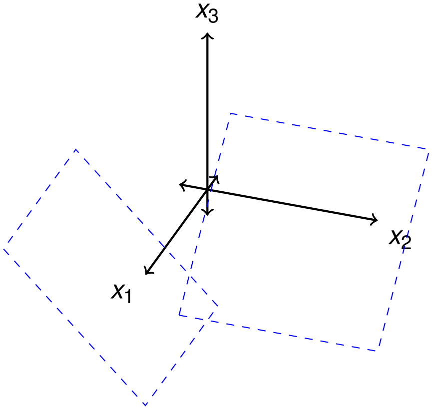
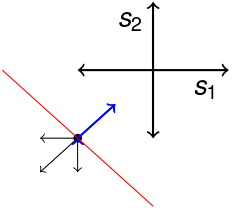


Coordinates $s = b - Ax$.

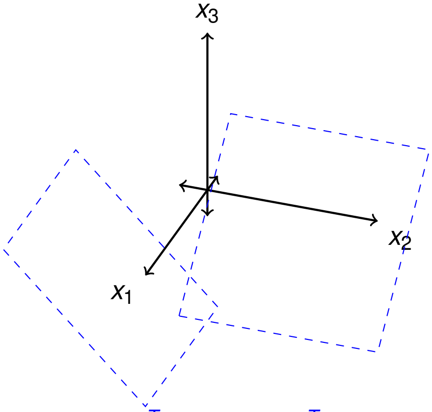
$x \geq 0$ where $s = 0$?



y where $y^T(b - Ax) < y^T(0) = 0$ for all $x \geq 0$ $\rightarrow$ $y^T b < 0$ and $y^T A \geq 0$.

$$Ax = b,\ x \geq 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$



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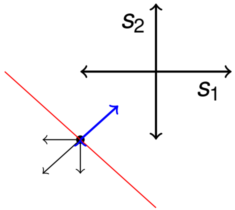

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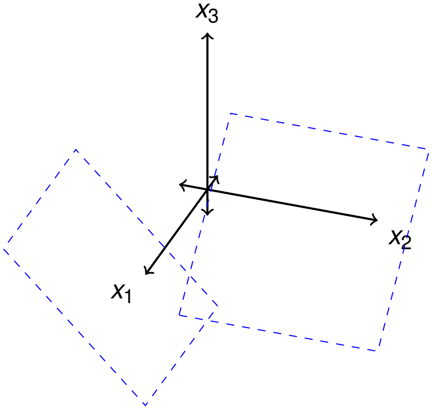

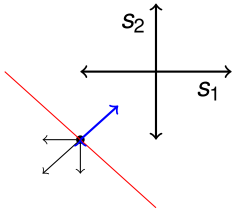

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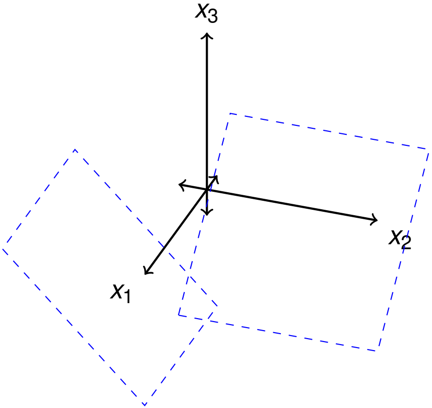

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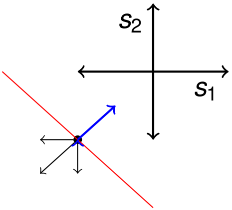

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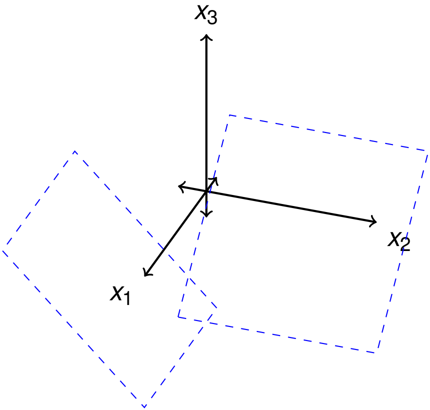

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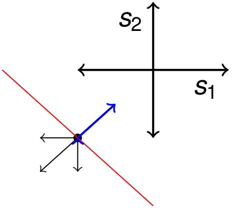

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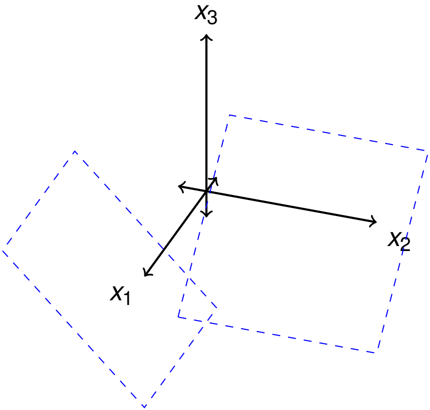

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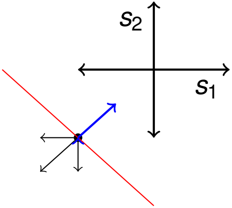

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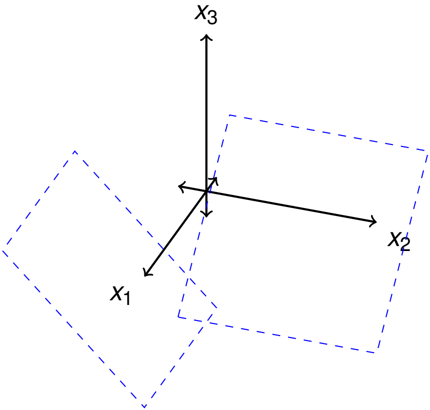

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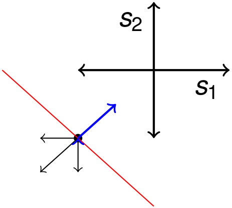

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Farkas 2

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Strong Duality

(From Goemans notes.)

Primal P $$z^* = \min c^T x$$
$$Ax = b$$
$$x \geq 0$$

Dual D $$: w^* = \max b^T y$$
$$A^T y \leq c$$

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Weak Duality: x, y- feasible P, D: $x^T c \geq b^T y$.

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$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 \end{aligned}$$

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Feasible $\tilde{x}$ for Primal.
(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.
(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible
$c^T(\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \to -\infty$ as $\mu \to \infty$

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, minimum value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want y where $\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}$. Let $A' = \begin{pmatrix} A^T \\ -b^T \end{pmatrix}$

Recall Farkas B: Either (1) $A'x' \leq b'$ or (2) $y'^T A' = 0, y'^T b' < 0, y' \geq 0$.

If (1) then done, otherwise (2) $\implies \exists y' = [x, \lambda] \geq 0$.

$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \qquad \begin{pmatrix} c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$.
Feasible $\tilde{x}$ for Primal.
(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.
(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible
$c^T(\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \to -\infty$ as $\mu \to \infty$
Primal unbounded!

Done

Today:

Done

Today:
Matching and simplex.

Done

Today:

Matching and simplex.

Convex separator.

Done

Today:
  Matching and simplex.
  Convex separator.
  Farkas.

Done

Today:
- Matching and simplex.
- Convex separator.
- Farkas.
- Strong Duality.

Done

Today:
  Matching and simplex.
  Convex separator.
  Farkas.
  Strong Duality.

Exercise:

Done

Today:
  Matching and simplex.
  Convex separator.
  Farkas.
  Strong Duality.

Exercise: Is there an algorithm there?

Done

Today:
  Matching and simplex.
  Convex separator.
  Farkas.
  Strong Duality.

Exercise: Is there an algorithm there?

See you on Thursday.