Linear Program.

How? From lecture warmup.

Linear program: $\max cx, Ax \le b, x \ge 0$
Dual: $\min y^T b, y^T A \ge c, y \ge 0$

Note: Dual variables correspond to primal equations and vice versa.

Weak Duality:

$y^T b \ge y^T Ax \ge cx$

First inequality from $b \ge Ax$ and second from $y^A \ge c$.

Complementary slackness:
(1) $x_j > 0 \implies a^{(j)} y = c_j$
(2) $y_i > 0 \implies a_i x = b_i$

What does multiplying by 0 do?
Zero and one. My love is won. Nothing and nothing done.

(2) $\implies y^T b = \sum_i y_i b_i = \sum_i y_i (a_i x) = y^T Ax$.

Similarly: (1) $\implies y^T Ax = cx$.

Complementary slackness conditions imply optimality.

Perfect Matching

Linear program: $\max \sum_e w_e x_e$, $\forall v : \sum_{e=(u,v)} x_e \le 1$, $x_e \ge 0$
$x_e = 1$ if $e \in M$, $x_e = 0$ otherwise. (Note: integer solution.)
Dual: $\min \sum_v p_v$, $\forall e = (u,v) : p_u + p_v \ge w_e$, $p_u \ge 0$.

Dual feasible at start: $p_u \ge \max_{e=(u,v)} w_e$
Maintain feasibility: adjust prices by δ.
Maintain Primal feasibility.
Maintain complementary slackness (2).
$x_e > 0$ only if $p_u + p_v = w_e$.
Eventually match all vertices.

The Engine that pulls the train:
Alternating path of tight edges.

Complementary slackness (1): Terminate when perfect matching.
$\forall v : \sum_{e=(u,v)} x_e = 1$. So any p_u can be non-zero.

The "play" indicates game playing.
Two person games: von Neuman. Equilibrium: Nash.

Is the path fundamental?
Are things as easy or as hard as $0, 1, 2, \ldots$...?

Duality.

Geometric View, Linear Equation, and Combinatorial.

Today: Strong Duality from Geometry.

Simplex Algorithm

$\max c \cdot x$.

$Ax \le b$
$x \ge 0$

Start at feasible point where n equations are satisfied.

E.g., $x = 0$.
This is a point.
Another view: intersection of n hyperplanes.

Drop one equation:
Points on line satisfy $n-1$ ind. equations.
Intersection of $n-1$ hyperplanes.

Move in direction that increases objective.

Until new tight constraint.

No direction increases objective.

Hyperplane View

$x + y + z \le 1$

On one side of hyperplane defined by $x + y + z = 1$.

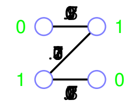
Normal to hyperplane? $(1,1,1)$.

Why? Normal: $u \cdot (v - w) = 0$ for any v, w in hyperplane.
(a,b,c) where $a + b + c = 1$.
(a', b', c') where $a' + b' + c' = 1$.
$(a' - a, b' - b, c' - c) \cdot (1,1,1) = (a' + b' + c' - (a + b + c)) = 0$.

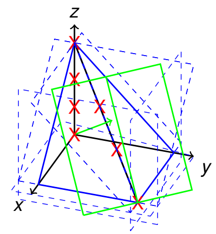
Normal to $mx + ny + pz = C$? (m,n,p)

Points in hyperplane are related by nullspace of row.

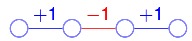
Maximum matching and simplex.



$$\begin{array}{rl}
\max x + y + z & \\
x \le 1 & \\
x + z \le 1 & a = 1 \\
z + y \le 1 & b = 1 \\
y \le 1 & \\
x \ge 0 & \\
y \ge 0 & \\
z \ge 0 & c = 1
\end{array}$$



Sum: $x + z + y$.



Augmenting Path. Via Gaussian Elimination!

Strong Duality

Convex Separator.

Farkas

Strong Duality!!!!! Maybe.

Linear Equations.

$Ax = b$

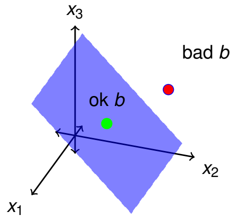
A is $n \times n$ matrix...

..has a solution.

If rows of A are linearly independent.
$y^T A \neq 0$ for any y

..or if b in subspace of columns of A.

If no solution, $y^T A = 0$ and $y \cdot b \neq 0$.



Convex body.

A a set of points P is *convex* if $x, y \in P$ implies that

$\alpha x + (1-\alpha) y \in P$

for $\alpha \in [0,1]$.

That is, the points in between x and y are in P.

Exercise:

$$Ax \leq b, x \geq 0$$

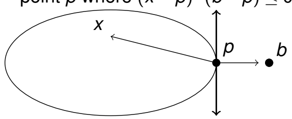
defines a convex set of points.

Convex Body and point.

For a convex body P and a point b,
either $b \in P$
or there is a hyperplane that separates P from b.

Separating hyperplace: v, where $v \cdot x < v \cdot b$, for all $x \in P$

point p where $(x-p)^T(b-p) \leq 0$



Take $v = (b-p)$.

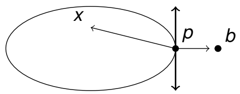
$(x \cdot v) = x \cdot (b-p) \leq p \cdot (b-p) = p \cdot v < b \cdot v$.
$p \cdot (b-p) < b \cdot (b-p)$?

$pb - p^2 < b^2 - bp$ iff $b^2 - 2pb + p^2 > 0$.

That is, if $(b-p)^2 > 0$. Is this always true?

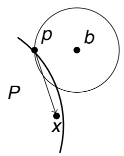
Proof.

For a convex body P and a point b,
either $b \in A$
or there is point p where $(x-p)^T(b-p) \leq 0 \; \forall x \in P$.



Proof: Choose p to be closest point to b in P.

Done or $\exists\, x \in P$ with $(x-p)^T(b-p) > 0$



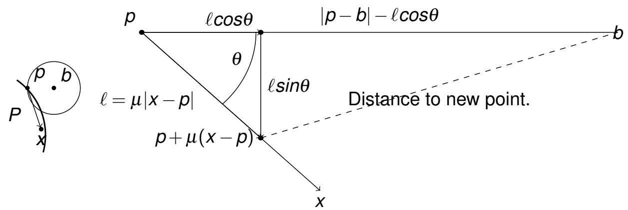
$(x-p)^T(b-p) \geq 0$
$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

Must be closer point b on line from p to x.

All points on line to x are in polytope.

Contradicts choice of p as closest point to b in polytope.

More formally.



Squared distance to b from $p + (x-p)\mu$
point between p and x
$(|p-b| - \mu|x-p|cos\theta)^2 + (\mu|x-p|sin\theta)^2$
θ is the angle between $x-p$ and $b-p$.

Simplify:
$|p-b|^2 - 2\mu|p-b||x-p|cos\theta + (\mu|x-p|)^2$.
Derivative with respect to μ ...
$-2|p-b||x-p|cos\theta + 2(\mu|x-p|^2)$.
which is negative for a small enough value of μ (for positive $cos\theta$.)

Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

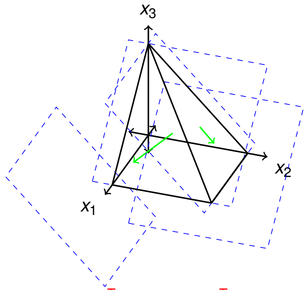
From $Ax = b$ use row reduction to get, e.g., $\hat{0} \cdot x = 0 \neq 5$.
That is, find y where $y^T A = \hat{0}$ and $y^T b \neq 0$.
Space is image of A. Affine subspace is columnspan of A.
y is normal. y in nullspace for column span.
$y^T b \neq 0 \implies b$ not in column span.

There is a separating hyperplane between any two convex bodies.

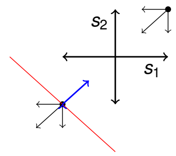
Idea: Let closest pair of points in two bodies define direction.

$Ax = b$, $x \geq 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$



Coordinates $s = b - Ax$.
$x \geq 0$ where $s = 0$?



y where $y^T(b - Ax) < y^T(0) = 0$ for all $x \geq 0$ $\rightarrow y^T b < 0$ and $y^T A \geq 0$.
Why? If $y \cdot A^{(i)} < 0$, then take $x_i \rightarrow \infty$, $y^T b - y^T Ax \rightarrow +\infty$,
Contradiction.

Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$ or (2) $y^T A \geq 0, y^T b < 0$.

Farkas 2

Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^T A \geq 0, y^T b < 0$.

Farkas B: Solution for exactly one of:
(1) $Ax \leq b$
(2) $y^T A = 0, y^T b < 0, y \geq 0$.

Strong Duality

(From Goemans notes.)

Primal P $\quad z^* = \min c^T x$
$Ax = b$
$x \geq 0$

Dual D $: w^* = \max b^T y$
$A^T y \leq c$

Weak Duality: x, y- feasible P, D: $x^T c \geq b^T y$.

$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 \end{aligned}$$

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, minimum value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want y where $\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}$. Let $A' = \begin{pmatrix} A^T \\ -b^T \end{pmatrix}$

Recall Farkas B: Either (1) $A'x' \leq b'$ or (2) $y'^T A' = 0, y'^T b' < 0, y' \geq 0$.

If (1) then done, otherwise (2) $\implies \exists y' = [x, \lambda] \geq 0$.

$(A \quad -b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \qquad (c^T \quad -z^*) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$.
Feasible $\tilde{x}$ for Primal.
(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.
(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible
$c^T(\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \rightarrow -\infty$ as $\mu \rightarrow \infty$
Primal unbounded!

Done

Today:
Matching and simplex.
Convex separator.
Farkas.
Strong Duality.

Exercise: Is there an algorithm there?

See you on Thursday.