Today

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Probability.

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Concentration.

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Randomized min-cut algorithm:
  Collapse a random edge repeatedly until two vertices.
  Report corresponding cut.

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Kawarabashi and Thorup (and...) give a near-linear proof and deterministic algorithm for this.

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An useful expression for $Pr[X \geq n/2 + \rho n/2]$?

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Generalized: Strictly increasing function, $g(\cdot)$. $Pr[X \geq x] \leq \frac{E[g(X)]}{g(x)}$.

Proof: "$X > x$" $\equiv$ "$g(X) \geq g(x)$".

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Coin flipping: For n fair coins, let X be number of heads:
$Pr[X \geq 3n/4] \leq \frac{n/2}{3n/4} = 2/3$.

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Chebyshev: $Pr[|X - E[X]| \geq x] \leq \frac{Var(X)}{x^2}$

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Also $Var(X) = E[X^2] - 2E[X]E[X] + E[X]^2 = E[X^2] - E[X]^2$.

The standard deviation $\delta(x) = \sqrt{Var(X)}$.

We want to know how far from average a variable is: $E[|X - E[X]|]$.

$Var(X) = E[(X - E[X])^2]$ in "nicer."
E.g. $Var(X + Y) = Var(X) + Var(Y)$ for independent random variables.

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Prove this convergence?

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Higher Moments?

Chebyshev was better than Markov, so...?

Thm: $Pr[|X - E[X]| \geq x] \leq \frac{E[(X-E[X])^k]}{x^k}$ for even k.

Coin Flips: $E[(X - E[X])]^4 = \frac{1}{16}(\binom{4}{2}\binom{n}{2} + \binom{n}{1})$.

Proof Idea:
Use $Y_i = 2X_i - 1$ to get ± 1 variables with mean zero.
Consider expansion $E[(\sum_i Y_i)^4]$.
Anything with odd power of Y_i has expectation 0.
Others have expectation 1. ..then count. □Yuck.

For coins: $Pr[X \geq 3/4n] = O(1/n^2)$.

In terms of std. dev: $Pr[X - \mu \geq t\sqrt{n}/2] \leq O(1/t^4)$

Versus Gaussians: $\leq e^{\frac{-t^2}{2}}$.

Worse constants. And kind of a mess.

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Some bounds: picking t.

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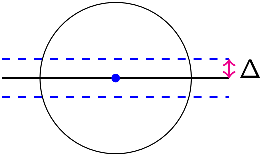

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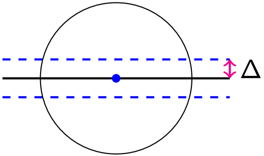

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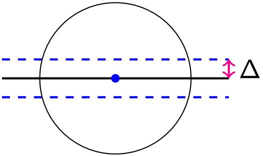

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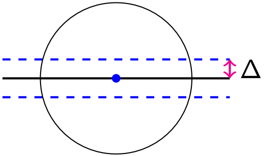

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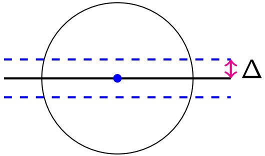

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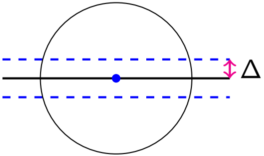

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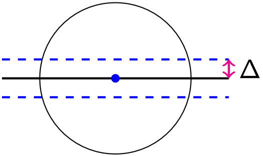

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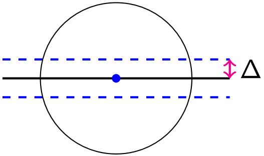

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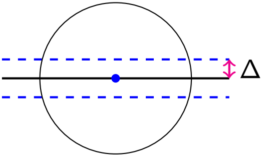

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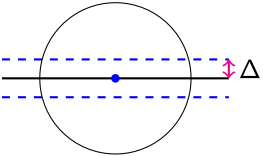

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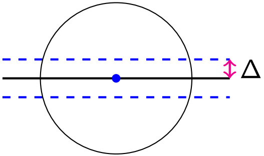

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Other views: KL distance.