Metric spaces.

A metric space X, $d(i,j)$ where
$d(i,j) \leq d(i,k) + d(k,j)$, $d(i,j) = d(j,i)$, $d(i,i) = 0$,
and $d(i,j) \geq 0$.

Which are metric spaces?

(A) X from R^d and $d(\cdot,\cdot)$ is Euclidean distance.

(B) X from R^d and $d(\cdot,\cdot)$ is squared Euclidean distance.

(C) X- vertices in graph, $d(i,j)$ is shortest path distances in graph.

(D) X is a set of vectors and $d(u,v)$ is $u \cdot v$.

(A) Obeys triangle inequality. (B) $a^2 + b^2 \leq (a+b)^2 = a^2 + 2ab + b^2$
(C) Shortest! (D) $1 \cdot -1 < 0$.

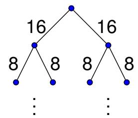
Input to TSP, facility location, some layout problems, ..., metric labelling.

Hard problems. Easier to solve on trees.
Dynamic programming on trees. Linear solving on trees.

Approximate metric on trees?

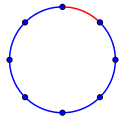
Approximate metric using a tree.

Tree metric:
X is nodes of tree with edge weights
$d_T(i,j)$ shortest path metric on tree.



Hierarchically well separated tree metric:
Tree weights are geometrically decreasing.

Probabilistic Tree embedding.
Map X into tree.
(i) No distance shrinks. (dominating)
(ii) Every distance stretches $\leq \alpha$
in expectation.



Distance 1 goes to $n-1$!
Bummer.

Map metric onto tree?

Fix it up chappie!

For cycle, remove a random edge get a tree.

Stretch of edge: $\frac{n-1}{n} \times 1 + \frac{1}{n} \times (n-1) \approx 2$
General metrics?

Probabilistic Tree embedding.

Probabilistic Tree embedding.
Map X into tree.
(i) No distance shrinks (dominating).
(ii) Every distance stretches $\leq \alpha$ in expecation.

Today: the tree will be Hierarchically well-separated (HST).
Elements of X are leaves of tree.

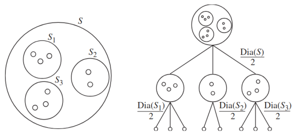
Useful: **use** spanning tree for graphical metrics.

The Idea:
HST $\equiv$ recursive decomposition of metric space.

Decompose space by diameter $\approx \Delta$ balls.
Recurse on each ball for $\Delta/2$.

Use randomness in
selection of ball centers.
the $\approx$ diameter of the balls.

Idea of decomposition.



Algorithm

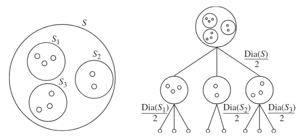
Algorithm: (X,d), $\text{diam}(X) \leq D$, $|X| = n$, $d(i,j) \geq 1$
1. π – random permutation of X.
2. Choose β in $[\frac{3}{8}, \frac{1}{2}]$ uniformly at random.

```
def subtree(S,Δ):
  T = []
  if Δ < 1 return [S]
  foreach i in π:
    if i ∈ S
      B = ball(i, βΔ) ; S = S/B
      T.append(B)
  return map (λ x: subtree(x,Δ/2), T);
```

3. subtree(X, D)

Tree has internal node for each level of call.
Tree edges have weight Δ to children.

Analysis: no distance shrinks.



Claim 1: $d_T(x,y) \geq d(x,y)$.

When $\Delta \leq d(x,y)$, x and y in diff. balls, $\implies$ cut at $\Delta \geq d(x,y)/2$.

$\rightarrow d_T(x,y) \geq \Delta + \Delta \geq d(x,y)$

Analysis: idea

Claim: $E[d_T(x,y)] = O(\log n) d(x,y)$.

Cut at level $\Delta \rightarrow d_T(x,y) \leq 4\Delta$. (Level of subtree call.)

Pr[cut at levelΔ]? Recall: cut at $\beta\Delta$, with $\beta \in [3/8, 1/2]$.

Would like $\leq \frac{d(x,y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta=D/2^i}(4\Delta)\frac{d(x,y)}{\Delta} = 4\log D \cdot d(x,y)$.

Why $\propto \frac{d(x,y)}{\Delta}$?
smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.
random diameter β shifts edge across $\Delta/8$.

$\rightarrow Pr[x,y$ cut by ball$|x$ in ball$] \approx \frac{d(x,y)}{\Delta/8}$

The problem?
Could be cut be many different balls.
For each probability is good, but could be hit by many.
random permutation to deal with this

Analysis: (x,y)

Would like $Pr[x,y$ cut by ball$|x$ in ball$] \leq \frac{8d(x,y)}{\Delta}$
(Only consider cut by x, factor 2 loss.)

At level Δ

At some point x is in some Δ level ball.
Renumber points in order of distance from x.

If $d(x,y) \geq \Delta/8$, $\frac{8d(x,y)}{\Delta} \geq 1$, so claim holds trivially.

Point j cuts (x,y) only if $d(j,x) \in [\frac{3\Delta}{8}, \frac{\Delta}{2}]$.
Call this set X_Δ.

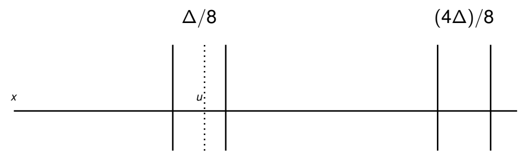
$j \in X_\Delta$ cuts (x,y) if..
$d(j,x) \leq \beta\Delta$ and $\beta\Delta \leq d(j,y) \leq d(j,x) + d(x,y)$
$\rightarrow \beta\Delta \in [d[j,x], d(j,x) + d(x,y)]$.
occurs with prob. $\leq \frac{d(x,y)}{\Delta/8} = \frac{8d(x,y)}{\Delta}$.

And j must be before any $i < j$ in $\pi \rightarrow$ prob is $\frac{1}{j}$

$\rightarrow Pr[j$ cuts $(x,y)] \leq \left(\frac{1}{j}\right)\frac{8d(x,y)}{\Delta}$

$d_T(x,y)$ at level Δ is 4Δ. $\rightarrow E[d_T(x,y)] = \sum_{\Delta=\frac{D}{2^i}}\sum_{j\in X_\Delta}\left(\frac{1}{j}\right)32d(x,y)$

A picture.



u is jth node in ball

captures x if first in π Probability of this is $1/j$.

Edge cut with probability $d(u,v)/(\Delta/8)$ due to $\beta \in [3/8, 1/2]$.

When Δ was 4Δ the nodes that capture x are separate.

The pipes are distinct!

$E(d_T(x,y)] = \sum_{\Delta=D/2^i}\sum_{j\in X_\Delta}\left(\frac{1}{j}\right)32d(x,y)$

Recall X_Δ has points with $d(x,j) \in [3\Delta/8, \Delta/2]$

"Listen Stash, the pipes are distinct!!"

Uh.. well X_Δ is distinct from $X_{\Delta/2}$.

$E(d_T(x,y)] = \sum_{\Delta=\frac{D}{2^i}}\sum_{j\in X_\Delta}\left(\frac{1}{j}\right)32d(x,y)$
$\leq \sum_j\left(\frac{1}{j}\right)32d(x,y)$
$\leq (32\ln n)(d(x,y))$.

Claim: $E[d_T(x,y)] = O(\log n)d(x,y)$

Expected stretch is $O(\log n)$.

We gave an algorithm that produces a distribution of trees.

The expected stretch of any pair is $O(\log n)$.

Metric Labelling

Input: graph $G = (V,E)$ with edge weights, $w(\cdot)$, metric labels (X,d), and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \rightarrow X$ that minimizes

$\sum_{e=(u,v)} c(e)d(l(u), l(v)) + \sum_v c(v, l(v))$

Idea: find HST for metric (X,d).

Solve the problem on a hierarchically well separated tree metric.

Kleinberg-Tardos: constant factor on uniform metric.

Hierarchically well separated tree, "geometric", constant factor.

$\rightarrow O(\log n)$ approximation.