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Is $\mu \geq 2/n^2$?

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What will be a good t?

We don't know. Try all possible thresholds ($n-1$ possibilities), and hope there is a t leading to a good cut!

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Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\mu = 1 - \lambda_2$, and $h(G) \le h(S) \le \sqrt{2(1 - \lambda_2)}$. Done!

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Let D be distribution over $S_1, \ldots, S_{n-1}$ from the above process.

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Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\mu}$

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$$\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] = \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}]$$

$$\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|)$$

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$|a \cdot b| \leq \|a\| \|b\|$, as $a \cdot b = \|a\| \|b\| cos(a, b)$

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Applying with $a, b \in \mathbb{R}^{n^2}$ with $a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}(|x_i| + |x_j|)$

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$$\begin{aligned}
\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V - S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\
&\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\
&= \frac{1}{2} a \cdot b \\
&\leq \frac{1}{2}\|a\|\|b\|
\end{aligned}$$

Simplify numerator.

Recall $\mu = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}(|x_i| + |x_j|)$

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$$\begin{aligned}
\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\mu}{n}\sum_{i,j}(x_i - x_j)^2 \\
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\|b\|^2 &= \sum_{i,j} M_{ij}(|x_i| + |x_j|)^2 \\
&\leq \sum_{i,j} M_{ij}(2x_i^2 + 2x_j^2) \\
&= 4\sum_i x_i^2
\end{aligned}$$

Put together.

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\mu}$

Put together.

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Numerator:

$$\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] = \leq \frac{1}{2}\|a\|\|b\|$$

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Combining:

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Put together.

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\mu}$

Numerator:

$$\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] = \leq \frac{1}{2}\|a\|\|b\|$$

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Thus $\exists S_i$ such that $h(S_i) \leq \sqrt{2\mu}$, which gives $h(G) \leq \sqrt{2(1-\lambda)}$ □

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Cheeger proof magically does this!

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A metric space X, $d(i,j)$ where
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Approximate metric on trees?

Approximate metric using a tree.

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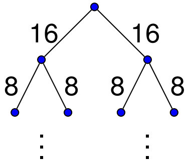
Tree weights are geometrically decreasing.



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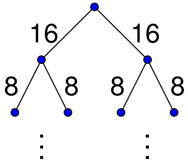

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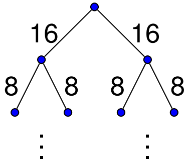

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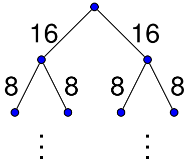

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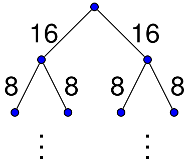

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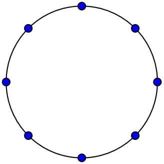

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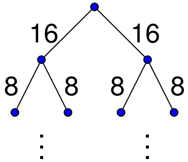

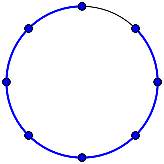

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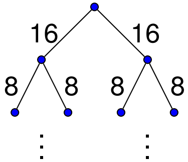

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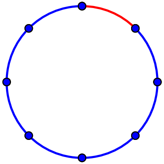

Distance 1 goes to $n-1$!

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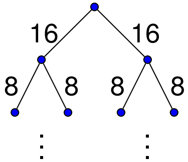

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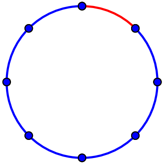

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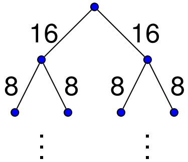

Probabilistic Tree embedding.
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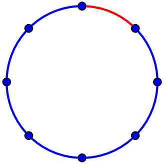

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Map metric onto tree?

Fix it up chappie!

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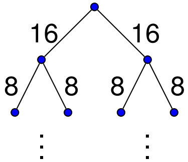

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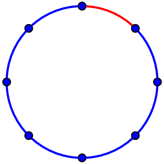

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For cycle, remove a random edge

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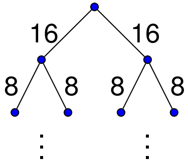

Hierarchically well separated tree metric:
 Tree weights are geometrically decreasing.

Probabilistic Tree embedding.
 Map X into tree.
 (i) No distance shrinks. (dominating)
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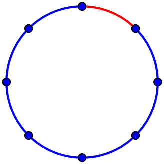

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Map metric onto tree?

Fix it up chappie!

For cycle, remove a random edge get a tree.

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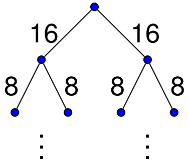

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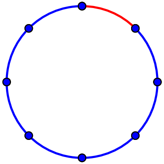

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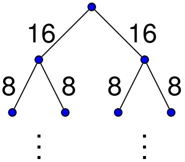

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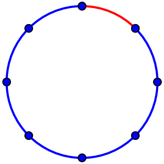

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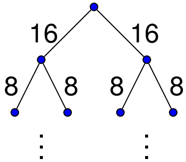

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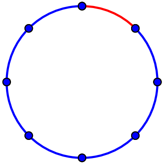

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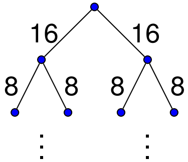

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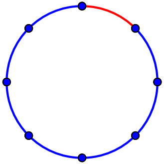

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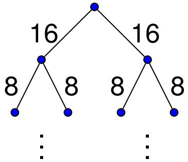

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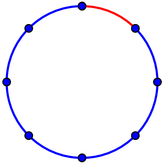

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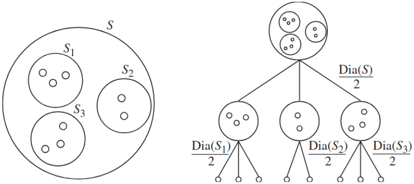
Idea of decomposition.



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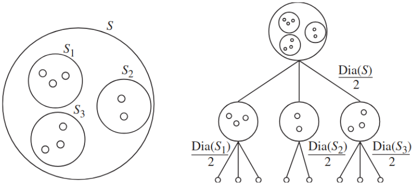

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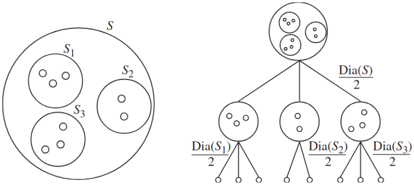

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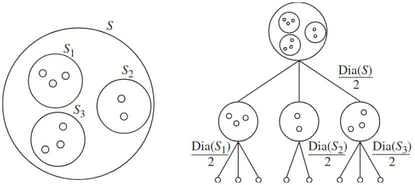

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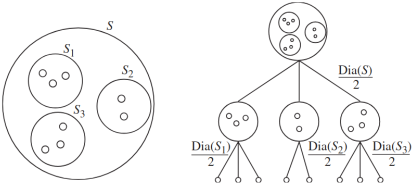

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random diameter β shifts edge across $\Delta/8$.

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$\rightarrow$ expected length is $\sum_{\Delta = D/2^i}(4\Delta)\frac{d(x,y)}{\Delta} = 4\log D \cdot d(x,y)$.

Why $\propto \frac{d(x,y)}{\Delta}$?

smaller the edge the less likely to be on edge of ball.

larger the delta, more room inside ball.

random diameter β shifts edge across $\Delta/8$.

$\rightarrow Pr[x,y$ cut by ball$|x$ in ball$] \approx \frac{d(x,y)}{\Delta/8}$

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Claim: $E[d_T(x,y)] = O(\log n) d(x,y)$.

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random permutation to deal with this

Analysis: (x, y)

Would like $Pr[x, y \text{ cut by ball} | x \text{ in ball}] \leq \frac{8d(x,y)}{\Delta}$

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A picture.

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u captures *x* if first out of *j* nodes in ball.

The pipes are distinct!

$$E(d_T(x,y)] = \sum_{\Delta=D/2^i} \sum_{j \in X_\Delta} \left(\frac{1}{j}\right) 32d(x,y)$$

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$$\leq (32\ln n)(d(x,y)).$$

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Claim: $E[d_T(x,y)] = O(\log n) d(x,y)$

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We gave an algorithm that produces a distribution of trees.

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"Listen Stash, the pipes are distinct!!"

Uh.. well X_Δ is distinct from $X_{\Delta/2}$.

$E(d_T(x,y)] = \sum_{\Delta=\frac{D}{2^i}} \sum_{j \in X_\Delta} \left(\frac{1}{j}\right) 32d(x,y)$
$\leq \sum_j \left(\frac{1}{j}\right) 32d(x,y)$
$\leq (32 \ln n)(d(x,y))$.

Claim: $E[d_T(x,y)] = O(\log n) d(x,y)$

Expected stretch is $O(\log n)$.

We gave an algorithm that produces a distribution of trees.

The expected stretch of any pair is $O(\log n)$.

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$\to O(\log n)$ approximation.