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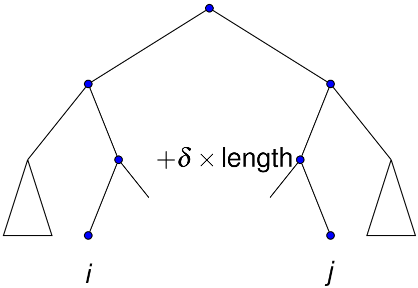

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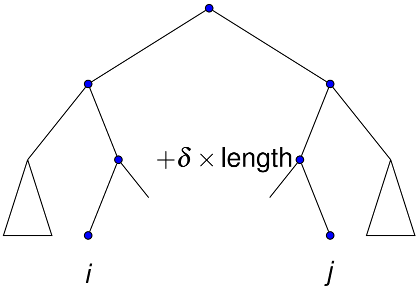

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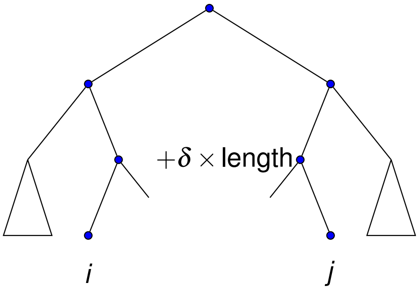

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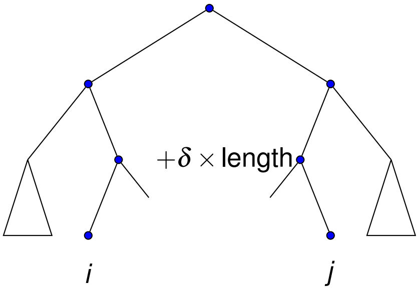

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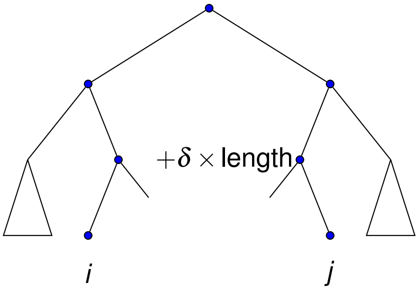

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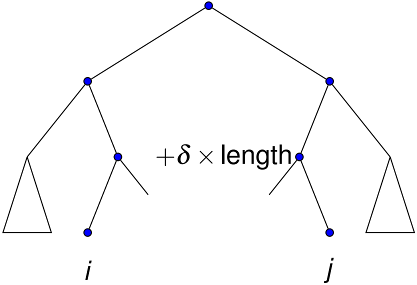

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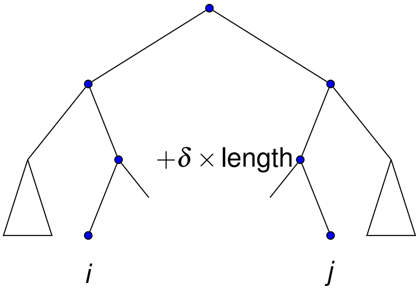

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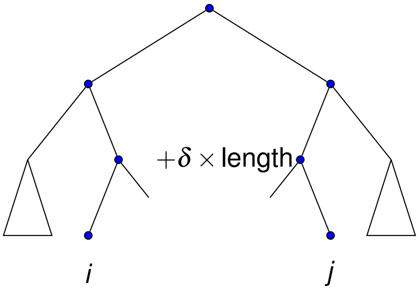

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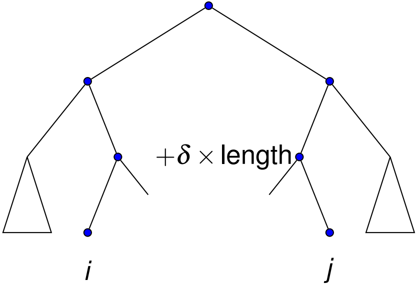

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$O(\log n)$ update/lookup.

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Given tree T.

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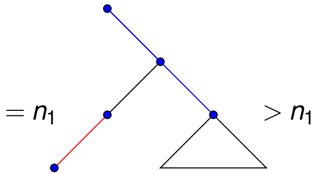

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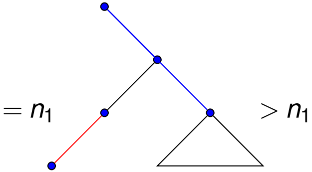

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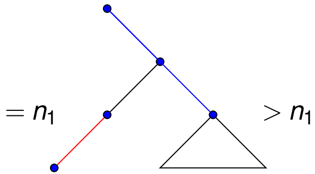

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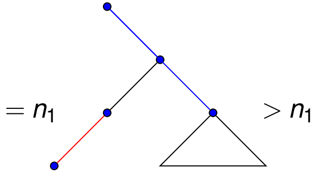

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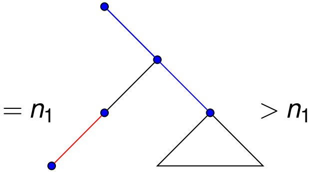

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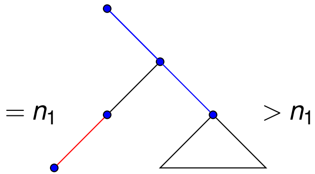

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$O(\log^2 n)$ update time for updating flow values on tree edges!

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Laplacian Systems are quite general:

Climate, physics, SDD-matrices.

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Today: the right hand inequality.

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Sum of squared differences.

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Alternatively: $\mu = \min_{x \perp 1} \frac{x^T (L/d) x}{x^t x}$.

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Rayleigh Quotient:

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Rayleigh Quotient:

embedding to line

that minimizes ratio of Average edge length/(avg pair length)

Hypercube

$V = \{0,1\}^d$

Hypercube

$V = \{0, 1\}^d \quad (x, y) \in E$

Hypercube

$V = \{0,1\}^d \quad (x,y) \in E$ when x and y differ in one bit.

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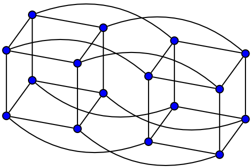

Good cuts?

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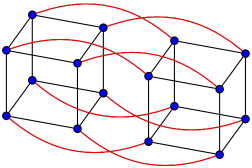

Good cuts? "Coordinate cut": d of them.

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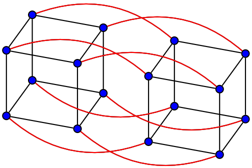

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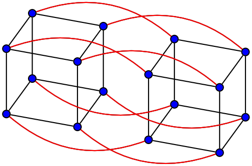

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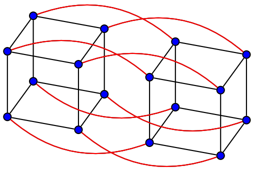

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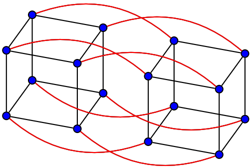

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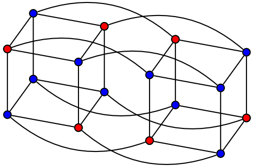

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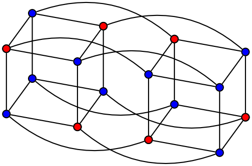

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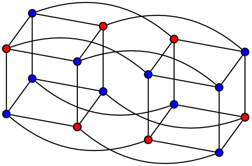

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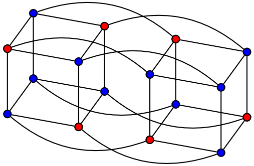

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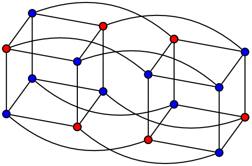

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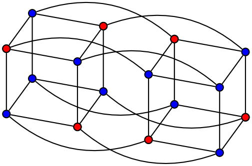

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Worse by a factor of $\sqrt{d}$

Eigenvalues of hypercube.

Anyone see any symmetry?

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Coordinate cuts. +1 on one side, -1 on other.

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Eigenvalue:

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Back to Cheeger.

Coordinate Cuts:

Back to Cheeger.

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Eigenvalue $1-2/d$.

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$\frac{\mu}{2}$

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Coordinate Cuts:
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$$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)}$$

Back to Cheeger.

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Back to Cheeger.

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Left hand side is tight.

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Eigenvector v maps to line.

Back to Cheeger.

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Eigenvector v maps to line.
Cut along line.

Back to Cheeger.

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Eigenvector algorithm gets a linear combination of coordinate cuts.

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Cycle

Tight example for Other side of Cheeger?

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Cycle on n nodes.

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Show eigenvalue gap $\mu \leq \frac{1}{n^2}$.

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Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T Mx}{x^T x}$ close to 1.

Slow vector.

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$$x_i = \begin{cases} i - n/4 & \text{if } i \leq n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$

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Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^TMx}{x^Tx}$ close to 1.

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Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

Eigenvalues of cycle?

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$x_i = \cos \frac{2\pi k i}{n}$

Eigenvalues of cycle?

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$x_i = \cos\frac{2\pi k i}{n}$

$(Mx)_i$

Eigenvalues of cycle?

Eigenvalues: $\cos\frac{2\pi k}{n}$.

$x_i = \cos\frac{2\pi k i}{n}$

$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right)$

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

$x_i = \cos \frac{2\pi k i}{n}$

$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right) = 2\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi k i}{n}\right)$

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Cuz: $\cos(x+y) + cos(x-y) = 2\cos x \cos y$.

Eigenvalues of cycle?

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Cuz: $\cos(x+y) + cos(x-y) = 2\cos x \cos y$.
cuz: $e^{i(x+y)} = e^{ix}e^{iy}$.

Eigenvalues of cycle?

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Eigenvalue: $\cos\frac{2\pi k}{n}$.

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Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues: vibration modes of system.

Eigenvalues of cycle?

Eigenvalues: $\cos\frac{2\pi k}{n}$.

$x_i = \cos\frac{2\pi ki}{n}$

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Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues: vibration modes of system. Fourier basis.

Eigenvalues of cycle?

Eigenvalues: $\cos\frac{2\pi k}{n}$.

$x_i = \cos\frac{2\pi ki}{n}$

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Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues: vibration modes of system. Fourier basis.

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Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues: vibration modes of system.   Fourier basis.

$\cos x \approx 1 + \sin(0)x - \cos(0)\frac{x^2}{2}$.

Eigenvalues of cycle?

Eigenvalues: $\cos\frac{2\pi k}{n}$.

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Cuz: $\cos(x+y) + cos(x-y) = 2\cos x \cos y$.
cuz: $e^{i(x+y)} = e^{ix}e^{iy}$. $e^{ix} = \cos x + i\sin x$. ....

Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues: vibration modes of system. Fourier basis.

$\cos x \approx 1 + \sin(0)x - \cos(0)\frac{x^2}{2}$.

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Eigenvalues of cycle?

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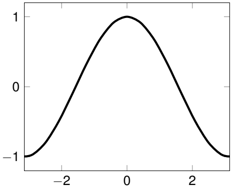

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Second biggest eigenvalue: $\approx 1 - \frac{2\pi^2}{n^2}$.

$h(G) = \frac{2}{n} \leq \frac{\pi}{n} \approx \sqrt{2\mu}$.

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p - probability distribution.

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Probability distrubtion after choose a random neighbor.

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Recall drunken sailor.

Other views

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Or use eigenvalue to upper bound on cut size.

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Or use eigenvalue to upper bound on cut size.
Is $\mu \geq 2/n^2$?

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Ratio of average edge “length” to pair length.

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Idea: pick random threshold.

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Ratio of average edge "length" to pair length.

Conductance $\phi(G) = n \frac{E(S, \overline{S})}{|S||V - S|}$.

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Interpretation of quadratic form

Consider $\mu = \min_{x \perp 1} \frac{x^T L x}{d x^T x}$.

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edges go a constant fraction of what random pairs do.

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What will be a good t?

We don't know. Try all possible thresholds ($n-1$ possibilities), and hope there is a t leading to a good cut!

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Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\mu = 1 - \lambda_2$, and $h(G) \leq h(S) \leq \sqrt{2(1 - \lambda_2)}$. Done!

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$$\exists S \qquad \tfrac{1}{d}|E(S, V - S)| - \sqrt{2\mu}\min(|S|, |V - S|) \le 0$$

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Let D be distribution over $S_1, \ldots, S_{n-1}$ from the above process.

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Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\mu}$

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$$\begin{aligned} \mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\ &\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \end{aligned}$$

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Applying with $a, b \in \mathbb{R}^{n^2}$ with $a_{ij} = \sqrt{M_{ij}} |x_i - x_j|, b_{ij} = \sqrt{M_{ij}} |x_i| + |x_j|$

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Numerator:

$$\begin{aligned}
\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\
&\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\
&= \frac{1}{2} a \cdot b \\
&\leq \frac{1}{2}\|a\|\|b\|
\end{aligned}$$

Simplify numerator.

Recall $\mu = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}(|x_i| + |x_j|)$

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\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\mu}{n}\sum_{i,j}(x_i - x_j)^2 \\
&= \frac{\mu}{n}\sum_{i,j}(x_i^2 + x_j^2) - \sum_{i,j} 2x_i x_j \\
&= \frac{\mu}{n}\sum_{i,j}(x_i^2 + x_j^2) - 2(\sum_i x_i)^2 \\
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&\le \sum_{i,j} M_{ij}(2x_i^2 + 2x_j^2) \\
&= 4\sum_i x_i^2
\end{aligned}
$$

Put together.

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\mu}$

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Thus $\exists S_i$ such that $h(S_i) \leq \sqrt{2\mu}$, which gives $h(G) \leq \sqrt{2(1-\lambda)}$ □

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Cheeger proof magically does this!