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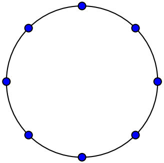

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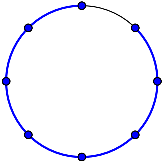

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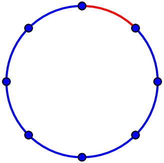

Distance 1 goes to $n-1$!

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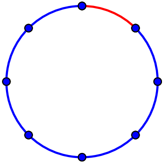

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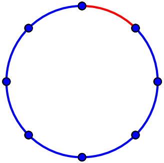

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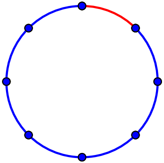

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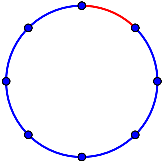

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Average Stretch of edge: $\frac{n-1}{n} \times 1$

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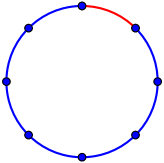

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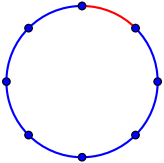

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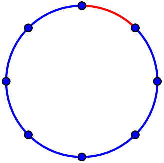

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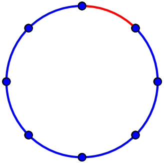

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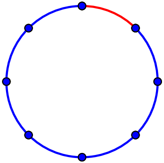

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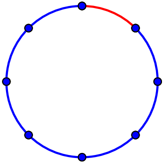

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Solve $\phi(u)$ and currents, $f(\cdot)$, on spanning tree.

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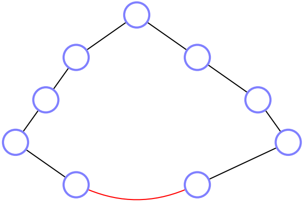

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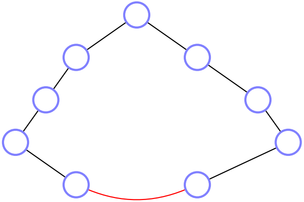

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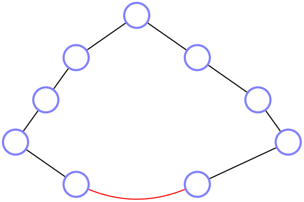

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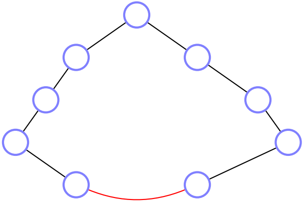

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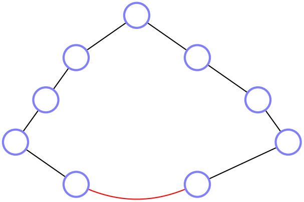

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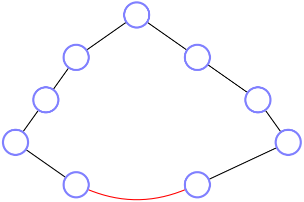

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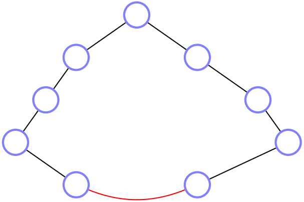

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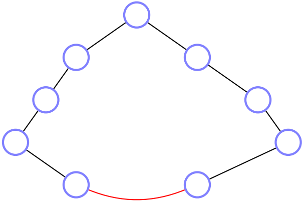

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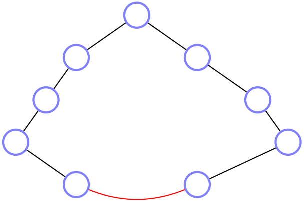

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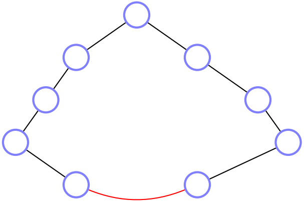

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Total violation: $\sum_{e=(u,v)}(f(e)-\phi(u,v))^2$.

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Update efficiently? Update on paths, path decomposition of trees.

Poll

What happens on the cycle?

Poll

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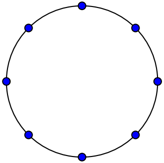

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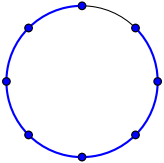

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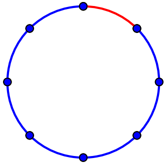

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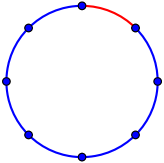

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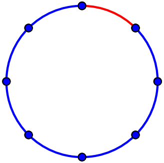

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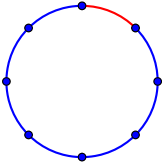

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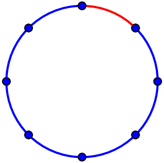

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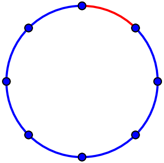

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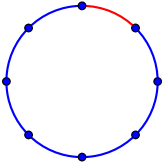

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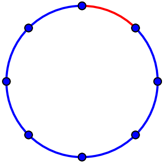

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$O(\log n)$ time per update.

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A graph $G = (V, E)$.

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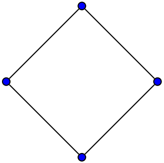

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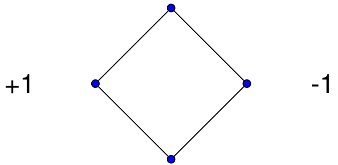

Claim: Minimizer is electrical flow.

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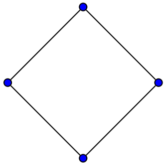
+1  -1

Claim: Minimizer is electrical flow.

Flow corresponds to flow induced by a set of potentials.

Some Matrices.

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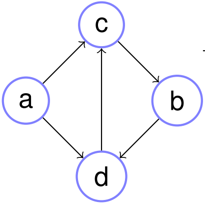

B

	a	b	c	d
(a,b)	1	-1	0	0
(a,c)	1	0	-1	0
(c,d)	0	0	1	-1
(d,b)	0	-1	0	1
(b,c)	0	1	-1	1

L

	a	b	c	d
a	2	-1	-1	0
b	-1	2	0	-1
c	-1	-1	3	-1
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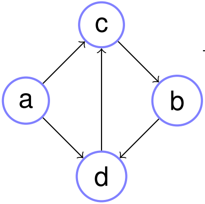

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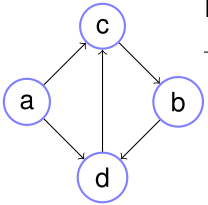

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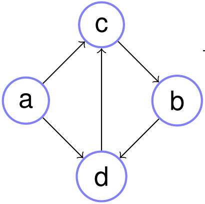

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(c,d)	0	0	1	-1
(d,b)	0	-1	0	1
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L

	a	b	c	d
a	2	-1	-1	0
b	-1	2	0	-1
c	-1	-1	3	-1
d	-1	-1	-1	3

Fun facts: $\mathbf{f} \in \Re^{|E|}$

$[B^T f]_u = \sum_{e=(u,v)} f_e - \sum_{e=(v,u)} f_e$

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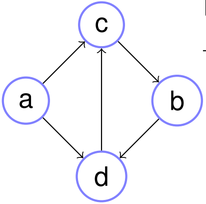
Some Matrices.

Given $G = (V,E)$, arbitrarily orient edges.

$$B_{v,e} = \begin{cases} -1 & e=(u,v) \\ 1 & e=(v,u) \\ 0 & \text{otherwise} \end{cases} \qquad L_{u,v} = \begin{cases} d & u=v \\ -1 & (u,v)\in E \\ 0 & \text{otherwise} \end{cases}$$



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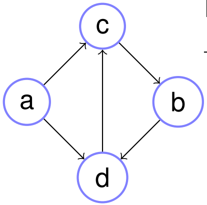

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Given G, χ, $\chi \perp 1$

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Minimize Squared Potential differences while routing current!

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Algorithm: Work on flow and potentials.
To drive gap to 0.

Alg.

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Take a spanning tree T of G. (Which tree?)

Route flow, f, to satisfy χ through T
Compute, ϕ, using tree ; $\phi_s = 0$, add f_e through T

Repeat:
Choose non-tree edge $e = (u, v)$ (Which non-tree edge?)
$f(e) = (\phi_u - \phi_v)/(\ell_T(u, v) + 1)$
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Route excess on path through tree.

Which Tree?
Claim: Linear time algorithm for T w/ stretch $O(m \log n \log \log n)$!
Stretch: $\sum_{e=(u,v)} \ell_T(u, v)$

Which non-tree edge?
Choose an edge w/prob. proportional to $\ell_T(e)$.

Finds $(1 + \varepsilon)$ approximation in $O(m \log n \log \log n \log(\frac{n}{\varepsilon}))$! ! !

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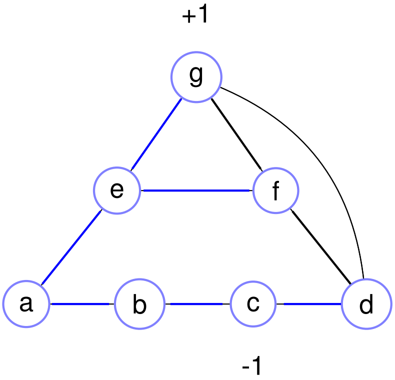

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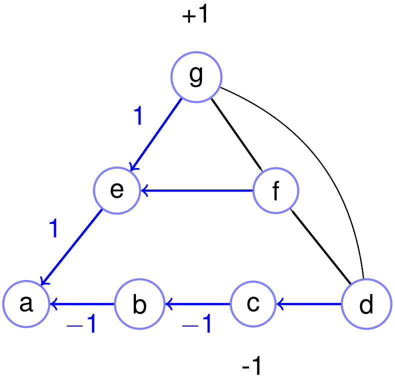

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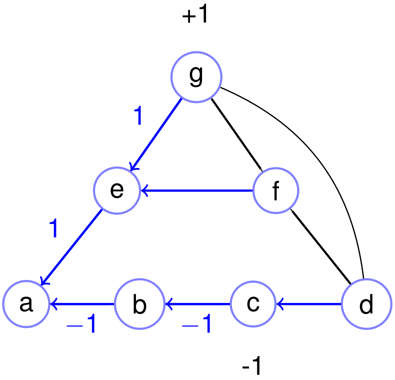

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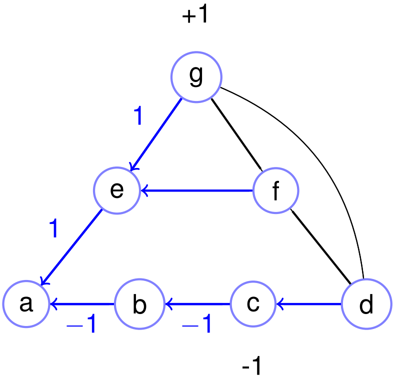

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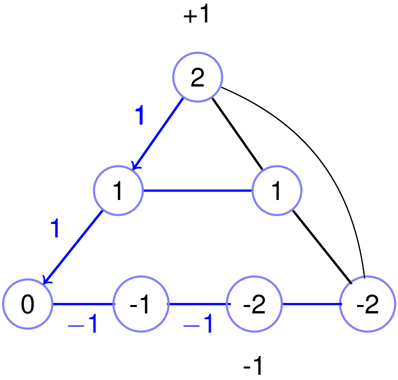

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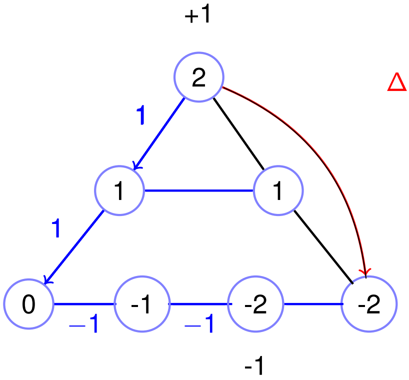

$\Delta_c = 4, R_e = 6$

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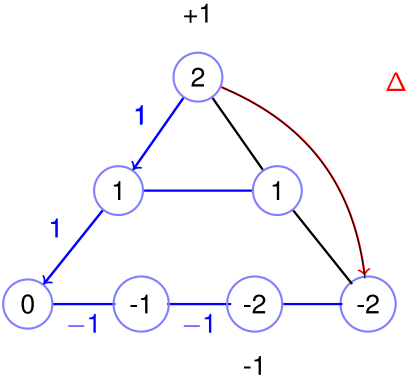

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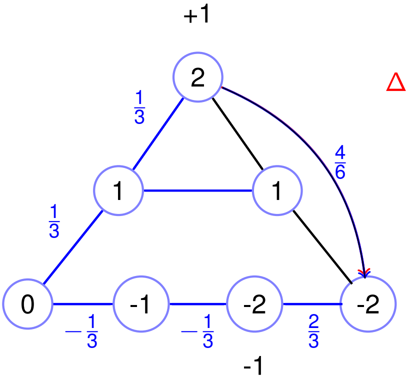

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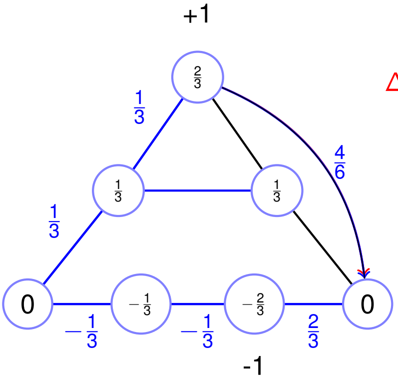

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Energy reduction.

Given T,$e = (u, v)$, let $R_e = \ell_T(u, v) + 1$.

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$\sum_{e'\in C_e}(f(e')-\delta)^2-(f(e'))^2=\sum_{e'\in C_e}-2f(e')\delta+\delta^2$

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Fix $1/R_e$ of a cycle violation!

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Algorithm maintains feasible ϕ, f, $(B^T f = \chi)$

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Total distance from optimal is cycle violations!

Polishing off.

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Duality Gap reduces by $(1 - 1/\tau)$ every iteration on expectation.

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Decompose tree into paths.

Interval Tree

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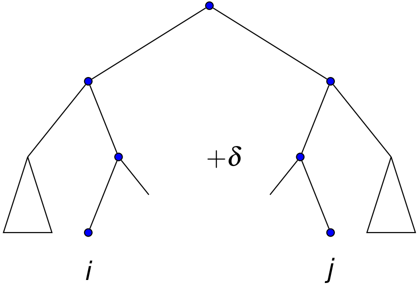

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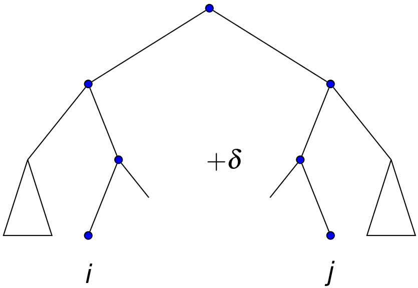

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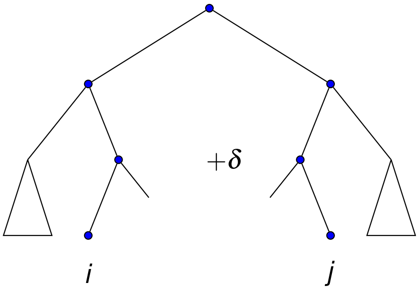

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For left path:
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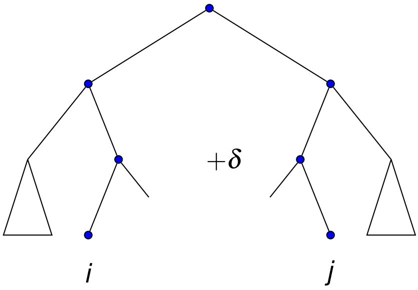

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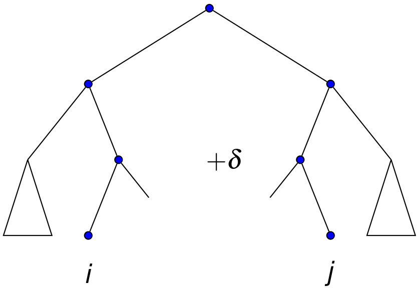

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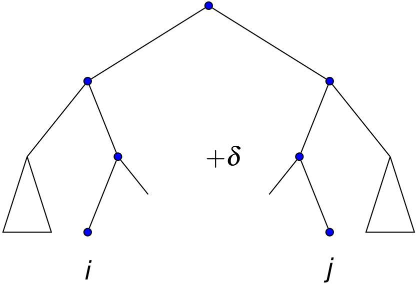

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Also, need number of descendants.

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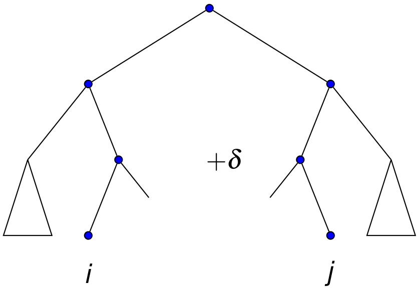

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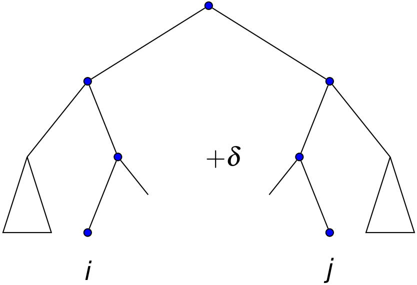

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$O(\log n)$ update/lookup.

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Given tree T.

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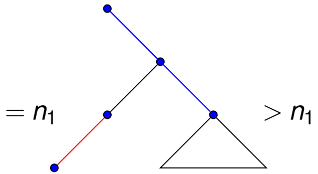

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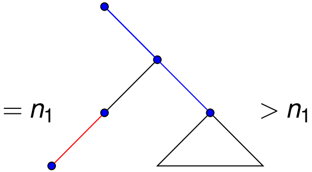

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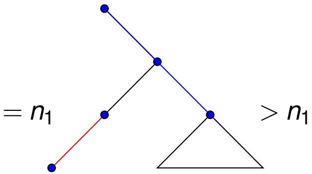

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Where a root to leaf path in tree only sees $O(\log n)$ paths.

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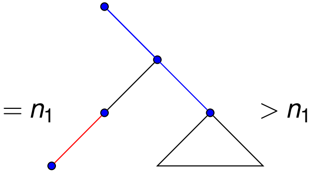

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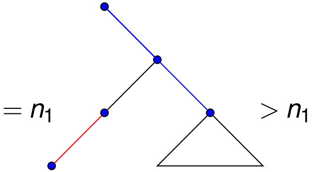

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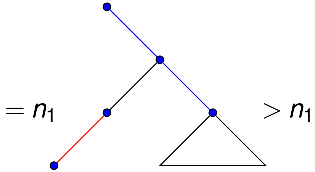

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$O(\log^2 n)$ update time for updating flow values on tree edges!

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Climate, physics, SDD-matrices.

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Today: the right hand inequality.

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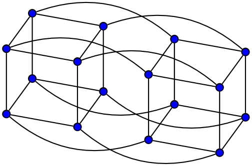

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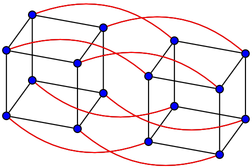

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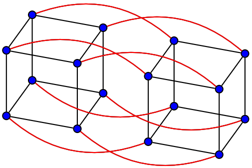

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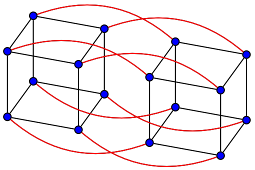

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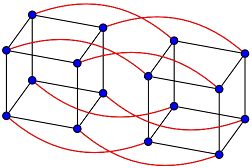

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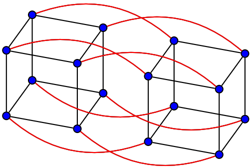

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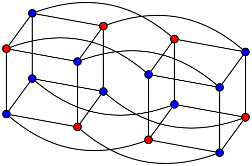

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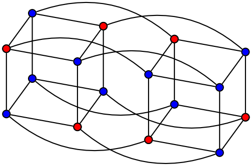

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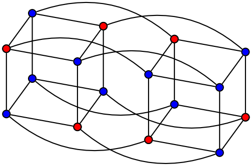

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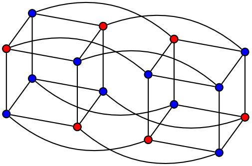

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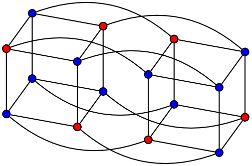

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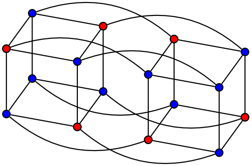

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Worse by a factor of $\sqrt{d}$

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Back to Cheeger.

Coordinate Cuts:

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For hypercube: $h(G) = \frac{1}{d}$ $\lambda_1 - \lambda_2 = 2/d$.
Left hand side is tight.

Note: hamming weight vector also in first eigenspace.

Lose "names" in hypercube, find coordinate cut?

Back to Cheeger.

Coordinate Cuts:
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Cut along line.

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Cycle

Tight example for Other side of Cheeger?

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Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T M x}{x^T x}$ close to 1.

Slow vector.

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Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

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$(Mx)_i$

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$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right)$

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$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right) = 2\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi k i}{n}\right)$

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Eigenvalues:
vibration modes of system.

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Fourier basis.

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Eigenvalues:
- vibration modes of system.
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$\cos x \approx 1 + \sin(0)x - \cos(0)\frac{x^2}{2}$.

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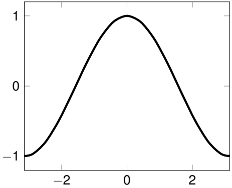

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$\cos \frac{2\pi}{n} \approx 1 + \sin(0)x - \cos(0)\frac{x^2}{2}$.

Second biggest eigenvalue: $\approx 1 - \frac{2\pi^2}{n^2}$.

$h(G) = \frac{2}{n} \leq \frac{\pi}{n} \approx \sqrt{2\mu}$.

Random Walk.

p - probability distribution.

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Probability distrubtion after choose a random neighbor.

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Recall drunken sailor.

Other views

Quotient Rayleigh: $\max_{x \perp 1} \frac{x^T M x}{x^t x}$.

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Cheeger Hard Part.

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Idea: We have $1-\lambda_2$ as a continuous relaxation of $\phi(G)$

Take the 2^{nd} eigenvector $x = \text{argmin}_{x \in \mathbb{R}^V - \text{Span}\{\mathbf{1}\}} \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}$

Consider x as an embedding of the vertices to the real line.

Round x to get a $x \in \{0,1\}^V$

Rounding: Take a threshold t,

$$\begin{cases} x_i \geq t & \rightarrow x_i = 1 \\ x_i < t & \rightarrow x_i = 0 \end{cases}$$

What will be a good t?

We don't know. Try all possible thresholds ($n-1$ possibilities), and hope there is a t leading to a good cut!

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Main Lemma: $G = (V, E)$, d-regular

$$x \in \mathbb{R}^V, x \perp \mathbf{1}, \mu = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{d}{n}\sum_{i,j}(x_i - x_j)^2}$$

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Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\mu=1-\lambda_2$, and $h(G)\le h(S)\le\sqrt{2(1-\lambda_2)}$. Done!

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Let D be distribution over $S_1, \ldots, S_{n-1}$ from the above process.

Denominator.

Goal: $\frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V-S)|]}{\mathbb{E}_{S \sim D}[min(|S|, |V-S|)]} \leq \sqrt{2\mu}$

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&= \sum_i \mathbb{E}_{S\sim D}[T_i] \\
&= \sum_i x_i^2
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&\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\
&= \frac{1}{2} a \cdot b \\
&\leq \frac{1}{2}\|a\|\|b\|
\end{aligned}
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Simplify numerator.

Recall $\mu = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}(|x_i| + |x_j|)$

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\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\mu}{n}\sum_{i,j}(x_i - x_j)^2 \\
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&= 4\sum_i x_i^2
\end{aligned}$$

Put together.

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Thus $\exists S_i$ such that $h(S_i) \leq \sqrt{2\mu}$, which gives $h(G) \leq \sqrt{2(1-\lambda)}$ $\square$

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Cheeger proof magically does this!