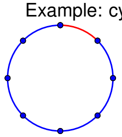
Low stretch trees.

Another cool approach:
Low stretch spanning tree.
Spanning tree T with small $\ell_T(u,v)$ for average edge $e=(u,v)$.
Example: expander. Any short tree. Diameter is $O(\log n)$.
Example: cycle.



Distance 1 goes to $n-1$!

For cycle, remove a random edge get a tree.

Average Stretch of edge: $\frac{n-1}{n} \times 1 + \frac{1}{n} \times (n-1) \approx 2$

In general: $\tilde{O}(m) = O(m \log n \log\log n)$.

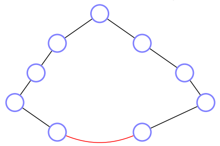
KOSZ

Laplacian systems $\equiv$ to routing electrical flow with sources and sinks.

Solve $\phi(u)$ and currents, $f(\cdot)$, on spanning tree.

$f(e) = (\phi(u)-\phi(v))/r(e) = \phi(u)-\phi(v) = \phi(u,v)$.
(Assume resistances $r(e)=1$.)

Pick non-tree edge with probability $\propto \ell_T(e)+1$



$f(e) \neq \phi(u,v)$

Change flow by $\frac{f(u,v)-\phi(u,v)}{\ell_T(e)+1}$.
Reduces $\sum_{e=(u,v)}(f(u,v)-\phi(u,v))^2$
Increases a bit on $\ell_T(e)$ (no violation) edges)
reduces a lot on e since large violation

Decrease on one edge dominate increase on many edges?
No violation on many, large violation on one. Quadratic.
Increase: $\ell \times (\varepsilon^2)$, decrease $1^2-(1-\varepsilon)^2 = -2\varepsilon+\varepsilon^2$.
Change: $-2\varepsilon+(\ell+1)\varepsilon^2$. $\varepsilon = \frac{1}{\ell+1} \implies -\varepsilon$ decrease.

Analysis Idea

Total violation: $\sum_{e=(u,v)}(f(e)-\phi(u,v))^2$.

Edge w/stretch ℓ selected: reduce violation by $\approx \frac{1}{\ell}(f(e)-\phi(u,v))^2$.

Non-tree edge reduces $\propto \alpha(f(e)-\phi(u,v))^2$
tree edges increase $\propto \ell(\alpha^2(f(e)-\phi(u,v))^2$
$\alpha = \Theta(1/\ell) \implies$ reduction $\propto \frac{1}{\ell}(f(e)-\phi(u,v))^2$.

Selected $\propto \ell$.
$(f(e)-\phi(u,v))^2 \times \frac{1}{\ell} \times \ell$
$\implies$ expected reduction $\propto \sum_{e=(u,v)}(f(e)-\phi(u,v))^2$.

The constant of proportionality: $\sum_e(\ell_T(e)+1)$. $=\tilde{O}(m)$
Probabilities: $\approx \frac{\ell_e+1}{\sum_e(\ell_T(e)+1)}$

$(1-\tilde{\Omega}(1/m))$ expected reduction.

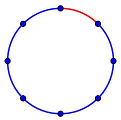
$\tilde{O}(m)$ iterations halves error.

Again: used exact solve on tree to speed up iterative method.
Shout out to "Algebra".

Update efficiently? Update on paths, path decomposition of trees.

Poll

What happens on the cycle?



Keep choosing non-tree edge. $\ell \approx m \implies \Omega(1/m)$ progress

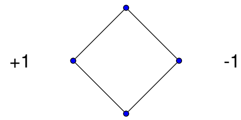
$\tilde{O}(m)$ iterations.

Need Data structure to update flow along path.
Exercise: Use Binary tree on path.
$O(\log n)$ time per update.

Electrical Flow and Laplacian Systems.

A graph $G=(V,E)$.
Circuit: nodes V, resistors E, value 1 (for today.)

Given $\chi : V \to \Re$
Find flow that routes χ and minimizes
$\sum_e f(e)^2$.



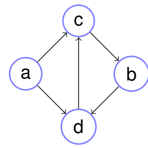
Claim: Minimizer is electrical flow.

Flow corresponds to flow induced by a set of potentials.

Some Matrices.

Given $G=(V,E)$, arbitrarily orient edges.

$$B_{v,e} = \begin{cases} -1 & e=(u,v) \\ 1 & e=(v,u) \\ 0 & \text{otherwise} \end{cases} \qquad L_{u,v} = \begin{cases} d & u=v \\ -1 & (u,v)\in E \\ 0 & \text{otherwise} \end{cases}$$



B

	a	b	c	d
(a,b)	1	-1	0	0
(a,c)	1	0	-1	0
(c,d)	0	0	1	-1
(d,b)	0	-1	0	1
(b,c)	0	1	-1	1

L

	a	b	c	d
a	2	-1	-1	0
b	-1	2	0	-1
c	-1	-1	3	-1
d	-1	-1	-1	3

Fun facts: $\mathbf{f} \in \Re^{|E|}$
$[B^T f]_u = \sum_{e=(u,v)} f_e - \sum_{e=(v,u)} f_e$
$B^T B = L$
$[Bx]_{e=(u,v)} = x_u - x_v$
$x^T L x = \sum_{e=(u,v)}(x_u-x_v)^2$

Duality..

Given G, χ, $\chi \perp 1$

Minimize $|f|^2$ subject to $B^T f = \chi$.

Lagrangian: $L(\phi, f) = \sum_e f(e)^2 + 2\phi^T(\chi - B^T f)$

Lagrangian Dual: Find ϕ that maximizes $\min_f L(\phi, f)$.

Given ϕ, minimize $L(\phi, f)$? Calculus.

For $e = (u, v)$
$2f(e) + 2(\phi_v - \phi_u) = 0$ (Minimum when partial derivatives = 0.)

$\rightarrow f(e) = (\phi_u - \phi_v)$ Potential differences!!!

Matrix Form: $f = B\phi$ Again, flows should be potential differences.

Dual problem: Find ϕ that maximizes ...
$\max L(\phi, B\phi) = \phi^T B^T B\phi + 2\phi^T(\chi - B^T B\phi) \implies \max_\phi 2\phi^T\chi - \phi^T L\phi$
Note: want $\phi^T L\phi = \sum_e (\phi_u - \phi_v)^2$ to be small.
Minimize Squared Potential differences while routing current!

Why did we take dual?

Dual problem:
Find ϕ that maximizes ...
$\max_\phi 2\phi\chi - \phi L\phi$

Take the derivative:
$L\phi - \chi$
$L\phi = \chi$ at optimal point!

Optimal potential is solution to a Laplacian linear system.

Also useful for convergence.

Algorithm maintains feasible ϕ, f,

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi^T L\phi$

Duality gap is "distance" from optimal!

Algorithm: Work on flow and potentials.
To drive gap to 0.

Alg.

Given: χ, G

Take a spanning tree T of G. (Which tree?)

Route flow, f, to satisfy χ through T
Compute, ϕ, using tree ; $\phi_s = 0$, add f_e through T

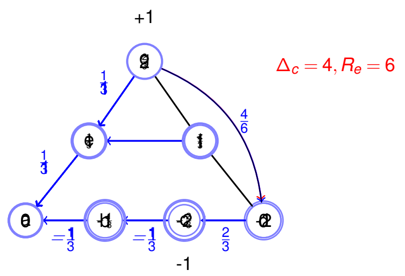
Repeat:
Choose non-tree edge $e = (u, v)$ (Which non-tree edge?)
$f(e) = (\phi_u - \phi_v)/(\ell_T(u, v) + 1)$
($\ell_T(u, v)$ path length in T)
Route excess on path through tree.

Which Tree?
Claim: Linear time algorithm for T w/ stretch $O(m \log n \log\log n)$!
Stretch: $\sum_{e=(u,v)} \ell_T(u, v)$

Which non-tree edge?
Choose an edge w/prob. proportional to $\ell_T(e)$.

Finds $(1+\varepsilon)$ approximation in $O(m \log n \log\log n \log(\frac{n}{\varepsilon}))$! ! ! ! ! !

Animation



Energy reduction.

Given T, $e = (u, v)$, let $R_e = \ell_T(u, v) + 1$.

Algorithm:

Repeatedly "Fix" edge $e = (u, v)$.

Route $-\delta = -\frac{\sum_{e' \in C_e} f(e')}{R_e}$ flow around cycle induced in T: C_e
(assume e' are oriented around cycle.)

Difference in energy from f and f'.
$\sum_{e' \in C_e} (f(e') - \delta)^2 - (f(e'))^2 = \sum_{e' \in C_e} -2f(e')\delta + \delta^2$
$= -(2\delta \sum_{e' \in C_e} f(e')) + R_E \delta^2$

Note: $\sum_{e' \in C_e} f(e') = R_e \delta$

$\rightarrow -\Delta^2_{C_e}/R_e$ where $\Delta_{C_e} = \sum_{e' \in C_e} f(e')$.

Fix a part of the potential difference, Δ_{C_e} around cyle!!

$\rightarrow$ reduction of $\Delta^2_{C_e}/R_e$ in energy!

Fix $1/R_e$ of a cycle violation!

Duality Gap?

Algorithm maintains feasible ϕ, f, $(B^T f = \chi)$

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi L\phi$

ϕ is tree induced voltages.

Total Duality Gap?

Gap: $|f|^2 - (2\phi^T\chi - \phi^T L\phi)$.

$= |f|^2 - 2\phi^T B^T f + \phi^T B^T B\phi$ where $B^T f = \chi$ and $L = B^T B$.
$= (f - B\phi)^T(f - B\phi)$.

Gap $= \sum_e (f(e) - \Delta_\phi(e))^2$ Difference between ϕ flow and f.

$\Delta_\phi(u, v) = \sum_{e \in P_{u,v}} -f(e)$. assume $f(e)$ is oriented around cycle.

For $e \in T$, $\Delta_\phi(e) = f(e)$. For $e \notin T$ $\Delta_{C_e} = f(e) + \sum_{e \in P_e} f(e)$

Duality Gap: $\sum_{e \notin T} (\Delta_{C_e}(f))^2$

Total distance from optimal is cycle violations!

Polishing off.

Claim: $E[\text{change in energy}|Gap] = \frac{Gap}{\tau}$
($\tau = \sum_e \ell_T(e)$ is stretch of E in T.)

Duality Gap: $\sum_{e \notin T} \Delta_{C_e}(f)^2$

Choose edge e reduce energy by $-\frac{\Delta_{C_e}^2}{R_e}$.

Choose edge with probability $\frac{R_e}{\tau}$.

Expected reduction $-\sum_e \frac{R_e}{\tau}\frac{\Delta_{C_e}^2}{R_e} = -\sum_e \frac{\Delta_{C_e}^2}{\tau}$ □

Duality Gap reduces by $(1-1/\tau)$ every iteration on expectation.

$O(\tau \log(n/\varepsilon))$ iterations gives $(1+\varepsilon)$ approximation.

$\tau = O(m \log n \log \log n)$ ...

$\tilde{O}(m)$ iterations

Iteration in $O(\log^2 n)$ time using balanced binary trees.

$\rightarrow \tilde{O}(m)$ time! ! ! ! ! !
Enough with the exclamations already !

Wrapup...

How to get low stretch tree?
Answer: Elkin-Spielman-Teng, ...,Abraham, Newman.
Open: Get $O(m \log n)$ stretch? Like for HSTs. (Later)
Ideas: Similar to HST construction, but much more subtle.

How to do update along cycle?
Answer: Data Structures.
Idea: Use a binary tree on paths.
Decompose tree into paths.

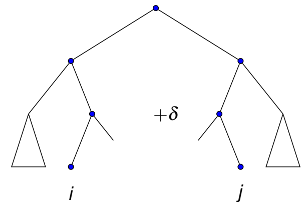
Interval Tree

Path: $G = (V,E)$, $V = \{1,\ldots,n\}$, $E = \{(i,i+1) : i \in \{1,n-1\}\}$

Init: $f(e) = 0$, $\forall e \in G$.

Update(i,j,δ)
$\forall k \in \{i,\ldots,j-1\}$, $f(i,i+1) = f(i,i+1) + \delta$

Lookup(i,j)
Return $\sum_{k=i}^{j-1} f(k,k+1)$.



Update: find common ancestor.
For left path:
Right children, add δ to edge.
Reflect for right path.

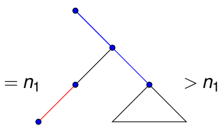
Lookup: Exercise.
Also, need number of descendants.

$O(\log n)$ update/lookup.

Path Decomposition of Tree

Given tree T.

Decomposition Procedure:
Longest path from root to leaf.
From root, go towards heavier side.
Remove. Recurse



Decomposition Property:
Where a root to leaf path in tree only sees $O(\log n)$ paths.

Proof Idea:
Every path change, doubles number of vertices.

$O(\log^2 n)$ update time for updating flow values on tree edges!

What's going on?

Geometric View.
Cycles are constraints.
Flow around cycle $= 0$.
Each cycle update is approximate projection
into subspace defined by constraint.
Solution is intersection of cycle constraints and flow conservation.
Kirchhoff's Laws.

Algorithmic Power.
Algebra: Solving exactly on tree.
Calculus: make a local move th decreases potential.

Better Algorithm:
Recursive algorithm give $O(m\sqrt{\log n})$ iterations to halve error.
Preconditioner uses tree to make sparse version of graph.

Correspondence to Practice:
Random sparsification of Cholesky factorization.
(Kyung-Sachdeva)
Tree pre-conditioner.

Laplacian Systems are quite general:
Climate, physics, SDD-matrices.

Next

Rayleigh quotient.
$\lambda_2 = \max_{x \perp 1} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2 = \frac{1}{d}\lambda_{min}(L)$.

Recall: $h(G) = \min_{S, |S| \le |V|/2} \frac{|E(S,V-S)|}{|S|}$

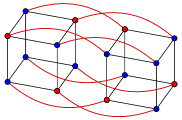
$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Today: the right hand inequality.

Hypercube

$V = \{0,1\}^d$ $(x,y) \in E$ when x and y differ in one bit.

$|V| = 2^d$ $|E| = d2^{d-1}$.



Good cuts? "Coordinate cut": d of them.
Edge expansion: $\frac{2^{d-1}}{d2^{d-1}} = \frac{1}{d}$

Ball cut: All nodes within $d/2$ of node, say $00\cdots0$.
Vertex cut size: $\binom{d}{d/2}$ bit strings with $d/2$ 1's.
$\approx \frac{2^d}{\sqrt{d}}$
Vertex expansion: $\approx \frac{1}{\sqrt{d}}$.
Edge expansion: $d/2$ edges to next level. $\approx \frac{1}{2\sqrt{d}}$

Worse by a factor of $\sqrt{d}$

Eigenvalues of hypercube.

Anyone see any symmetry?

Coordinate cuts. +1 on one side, -1 on other.

$(Mv)_i = (1-2/d)v_i$.

Eigenvalue $1-2/d$. d Eigenvectors. Why orthogonal?

Next eigenvectors?

Delete edges in two dimensions.

Four subcubes: bipartite. Color ± 1

Eigenvalue: $1-4/d$. $\binom{d}{2}$ eigenvectors.

Eigenvalues: $1-2k/d$. $\binom{d}{k}$ eigenvectors.

Back to Cheeger.

Coordinate Cuts:
Eigenvalue $1-2/d$. d Eigenvectors.

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

For hypercube: $h(G) = \frac{1}{d}$ $\lambda_1 - \lambda_2 = 2/d$.
Left hand side is tight.

Note: hamming weight vector also in first eigenspace.

Lose "names" in hypercube, find coordinate cut?

Find coordinate cut?

Eigenvector v maps to line.
Cut along line.

Eigenvector algorithm gets a linear combination of coordinate cuts.

Something like ball cut.

Find coordinate cut?

Cycle

Tight example for Other side of Cheeger?

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Cycle on n nodes.

Will show other side of Cheeger is tight.

Edge expansion:Cut in half.
$|S| = n/2$, $|E(S,\overline{S})| = 2$
$\to h(G) = \frac{2}{n}$.

Show eigenvalue gap $\mu \le \frac{1}{n^2}$.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^TMx}{x^Tx}$ close to 1.

Slow vector.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^TMx}{x^Tx}$ close to 1.

$$x_i = \begin{cases} i - n/4 & \text{if } i \le n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$

Hit with M.

$$(Mx)_i = \begin{cases} -n/4 + 1/2 & \text{if } i = 1, n \\ n/4 - 1 & \text{if } i = n/2 \\ x_i & \text{otherwise} \end{cases}$$

$\to x^TMx = x^Tx(1 - O(\frac{1}{n^2}))$ $\to \lambda_2 \ge 1 - O(\frac{1}{n^2})$

$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$

$h(G) = \frac{2}{n} = \Theta(\sqrt{\mu})$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

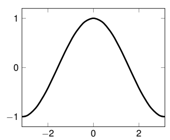
Eigenvalues: $\cos\frac{2\pi k}{n}$.

$x_i = \cos\frac{2\pi ki}{n}$

$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right) = 2\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi ki}{n}\right)$

Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues:
vibration modes of system.
Fourier basis.



$\cos x \approx 1 + \sin(0)x - \cos(0)\frac{x^2}{2}$.

$\cos\frac{2\pi}{n} \approx 1 + \sin(0)x - \cos(0)\frac{x^2}{2}$.

Second biggest eigenvalue: $\approx 1 - \frac{2\pi^2}{n^2}$.

$h(G) = \frac{2}{n} \le \frac{\pi}{n} \approx \sqrt{2\mu}$.

Random Walk.

p - probability distribution.

Probability distrubtion after choose a random neighbor.
Mp.

Converge to uniform distribution.

Power method: $M^t x$ goes to highest eigenvector.

$M^t x = a_1 \lambda_1^t v_1 + a_2 \lambda_2 v_2 + \cdots$

$\lambda_1 - \lambda_2$ - rate of convergence.

$\Omega(n^2)$ steps to get close to uniform.

Start at node 0, probability distribution, $[1,0,0,\cdots,0]$.
Takes $\Omega(n^2)$ to get n steps away.

Recall drunken sailor.

Interpretation of quadratic form

Consider $\mu = \min_{x \perp 1} \frac{x^T L x}{d x^T x}$.

$\sum_i x_i = 0$

Claim: $n \sum_i x_i^2 = \frac{1}{n} \sum_{i \le j} (x_i - x_j)^2$

$\sum_i (2(n-1)x_i^2 - 2x_i \sum_{j \neq i} x_j)$.

$-2x_i \sum_{j \neq i} x_j) = -2x_i((\sum_j x_j) - x_i) = 2x_i^2 + x_i \sum_i x_j = 2x_i^2$

$\sum_i n x_i^2 - (\sum_i x_i)(\sum_j x_j)$.

Claim: $x^L x = \sum_{e=(i,j)} M_{ij}(x_i - x_j)^2$

$\mu = \min_{x \perp 1} \frac{\sum_e M_{ij}(x_i - x_j)^2}{\frac{d}{n} \sum_{i,j} (x_i - x_j)^2}$

Ratio of average edge "length" to pair length.

Conductance $\phi(G) = n \frac{E(S,\bar{S})}{|S||V-S|}$.

Idea: pick random threshold.
Average number of edges cut compared to pairs is μ?

Other views

Quotient Rayleigh: $\max_{x \perp 1} \frac{x^T M x}{x^t x}$.

Alternatively: $\mu = \min_{x \perp 1} \frac{x^T (L/d) x}{x^t x}$.

$L = dI - A$ or $L/d = I - A/d = I - M$.

Also:

$x^T L x = \sum_i d x_i^2 - \sum_{e=(i,j)} 2x_i x_j$

$= \sum_{e=(i,j)} (x_i^2 + x_j^2 - 2x_i x_j)$

$= \sum_{e=(i,j)} (x_i - x_j)^2$.

Random threshold.

$\mu = \min_{x \perp 1} \frac{\sum_e M_{ij}(x_i - x_j)^2}{\frac{d}{n} \sum_{i,j} (x_i - x_j)^2}$

Conductance $\phi(G) = n \frac{E(S,\bar{S})}{|S||V-S|}$.

Idea: pick random threshold.

Average number of edges cut compared to pairs is μ.
True if "length" corresponded to $|x_i - x_j|$.
Off when length is $(x_i - x_j)^2$.

Off by average length of edges.
For path:
edges have $|x_i - x_j|$ being $1/n$ of that for pairs.
For expanders:
edges go a constant fraction of what random pairs do.

Warmup exercise: no larger eigenvalue on cycle.

Consider x for path.

Sort by x-value: $x_1, \ldots, x_n$.

Let $x_1^2 + x_n^2 = 1$. Shift so that $x_{n/2} = 0$.

Now, since any cut ≥ 1
$x^T L x \ge 1 \sum_i (x_i - x_{i+1})^2$.

Take $a = (x_i - x_{i+1})$ and $b = 1/\sqrt{n}$.

Cauchy-Schwartz $|a||b| \ge a \cdot b$.

Yields: $\sum_i (x_i - x_{i+1})^2 \ge (x_1 - x_n)^2/n$

Furthermore: $\sum_i x_i^2 \le n(x_1^2 + x_n^2) \le n(x_1 - x_n)^2$.

$\rightarrow \mu = \frac{x^T L x}{x^T x} \ge 1/n^2$.

Cheeger Hard Part.

Now let's get to the hard part of Cheeger $h(G) \le \sqrt{2(1-\lambda_2)}$.

Idea: We have $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$

Take the 2^{nd} eigenvector $x = \text{argmin}_{x \in \mathbb{R}^V - \text{Span}\{\mathbf{1}\}} \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$

Consider x as an embedding of the vertices to the real line.

Round x to get a $x \in \{0,1\}^V$

Rounding: Take a threshold t,

$\begin{cases} x_i \ge t & \rightarrow x_i = 1 \\ x_i < t & \rightarrow x_i = 0 \end{cases}$

What will be a good t?

We don't know. Try all possible thresholds ($n-1$ possibilities), and hope there is a t leading to a good cut!

Sweep Cut Algorithm

Input: $G = (V, E)$, $x \in \mathbb{R}^V, x \perp \mathbf{1}$

Sort the vertices in non-decreasing order with respect to x
WLOG $V = \{1, \dots, n\}$ $\quad x_1 \le x_2 \le \dots \le x_n$

Let $S_i = \{1, \dots, i\}$ $\quad i = 1, \dots, n-1$

Return $S = \text{argmin}_{S_i} h(S_i)$

Main Lemma: $G = (V, E)$, d-regular

$$x \in \mathbb{R}^V, x \perp \mathbf{1}, \mu = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{d}{n}\sum_{i,j}(x_i - x_j)^2}$$

If S is the ouput of the sweep cut algorithm, then $h(S) \le \sqrt{2\mu}$

Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\mu = 1 - \lambda_2$, and $h(G) \le h(S) \le \sqrt{2(1 - \lambda_2)}$. Done!

Proof of Main Lemma

WLOG $V = \{1, \dots, n\}$ $\quad x_1 \le x_2 \le \dots \le x_n$

Want to show

$$\exists i \text{ s.t. } h(S_i) = \frac{\frac{1}{d}|E(S_i, V - S_i)|}{\min(|S_i|, |V - S_i|)} \le \sqrt{2\mu}$$

Probabilistic Argument: Construct a distribution D over $\{S_1, \dots, S_{n-1}\}$ such that

$$\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V - S)|]}{\mathbb{E}_{S\sim D}[\min(|S|, |V - S|)]} \le \sqrt{2\mu}$$

$$\rightarrow \mathbb{E}_{S\sim D}[\tfrac{1}{d}|E(S, V - S)| - \sqrt{2\mu}\min(|S|, |V - S|)] \le 0$$

$$\exists S \quad \tfrac{1}{d}|E(S, V - S)| - \sqrt{2\mu}\min(|S|, |V - S|) \le 0$$

The distribution D

WLOG, shift and scale so that $x_{\lfloor \frac{n}{2} \rfloor} = 0$, and $x_1^2 + x_n^2 = 1$

Take t from the range $[x_1, x_n]$ with density function $f(t) = 2|t|$.

Check: $\int_{x_1}^{x_n} f(t)\mathrm{d}t = \int_{x_1}^{0} -2t\mathrm{d}t + \int_{0}^{x_n} 2t\mathrm{d}t = x_1^2 + x_n^2 = 1$

$S = \{i : x_i \le t\}$

Let D be distribution over $S_1, \dots, S_{n-1}$ from the above process.

Denominator.

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|, |V-S|)]} \le \sqrt{2\mu}$

Denominator:

Let T_i = indicator for "i is in the smaller set of $S, V - S$"

Can check

$$\mathbb{E}_{S\sim D}[T_i] = Pr[T_i = 1] = x_i^2$$

$$\begin{aligned}\mathbb{E}_{S\sim D}[min(|S|, |V - S|)] &= \mathbb{E}_{S\sim D}[\sum_i T_i] \\ &= \sum_i \mathbb{E}_{S\sim D}[T_i] \\ &= \sum_i x_i^2\end{aligned}$$

Numerator

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|, |V-S|)]} \le \sqrt{2\mu}$

Numerator:

Let $T_{i,j}$ = indicator for i, j is cut by $(S, V - S)$

$$\begin{cases} x_i, x_j \text{ same sign:} & Pr[T_{i,j} = 1] = |x_i^2 - x_j^2| \\ x_i, x_j \text{ different sign:} & Pr[T_{i,j} = 1] = x_i^2 + x_j^2 \end{cases}$$

A common upper bound: $\mathbb{E}[T_{i,j}] = Pr[T_{i,j} = 1] \le |x_i - x_j|(|x_i| + |x_j|)$

$$\begin{aligned}\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V - S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\ &\le \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|)\end{aligned}$$

Cauchy-Schwarz Inequality

$|a \cdot b| \le \|a\|\|b\|$, as $a \cdot b = \|a\|\|b\| cos(a, b)$

Applying with $a, b \in \mathbb{R}^{n^2}$ with $a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j|$

Numerator:

$$\begin{aligned}\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V - S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\ &\le \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\ &= \frac{1}{2} a \cdot b \\ &\le \frac{1}{2}\|a\|\|b\|\end{aligned}$$

Simplify numerator.

Recall $\mu = \frac{\sum_{i,j} M_{ij}(x_i-x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i-x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}(|x_i| + |x_j|)$

$$\begin{aligned}
\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\mu}{n}\sum_{i,j}(x_i - x_j)^2 \\
&= \frac{\mu}{n}\sum_{i,j}(x_i^2 + x_j^2) - \sum_{i,j} 2x_i x_j \\
&= \frac{\mu}{n}\sum_{i,j}(x_i^2 + x_j^2) - 2(\sum_i x_i)^2 \\
&\le \frac{\mu}{n}\sum_{i,j}(x_i^2 + x_j^2) = 2\mu \sum_i x_i^2
\end{aligned}$$

$$\begin{aligned}
\|b\|^2 &= \sum_{i,j} M_{ij}(|x_i| + |x_j|)^2 \\
&\le \sum_{i,j} M_{ij}(2x_i^2 + 2x_j^2) \\
&= 4\sum_i x_i^2
\end{aligned}$$

Put together.

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \le \sqrt{2\mu}$

Numerator:

$$\begin{aligned}
\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] &=\le \frac{1}{2}\|a\|\|b\| \\
&\le \frac{1}{2}\sqrt{2\mu\sum_i x_i^2}\sqrt{4\sum_i x_i^2} \quad = \sqrt{2\mu}\sum_i x_i^2
\end{aligned}$$

Recall **Denominator:**

$$\mathbb{E}_{S\sim D}[min(|S|,|V-S|)] = \sum_i x_i^2$$

We get

$$\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \le \sqrt{2\mu}$$

Thus $\exists S_i$ such that $h(S_i) \le \sqrt{2\mu}$, which gives $h(G) \le \sqrt{2(1-\lambda)}$ □

Kind of a proof.

$G = (V,E)$, $h = h(G)$.

Claim:
From $S \subset V$ of vertices, $|E(N_{1/h}(S))| \ge 2|E(S)|$.

Claim': there are $\Omega(h|S|)$ paths of length $\ell = 1/h$ in $N_{1/h}(S)$.

Cut size is $\ge h(G)|S| \implies$ flow of value $h(G)|S|$.
Max flow-min cut theorem.

From path argument: $\implies \mu \ge \frac{1}{\ell^2} = h(G)^2$.

Run argument over sets of size 2^i and one gets the upper bound.

Why no $\log n$ factor? The mass splits, and **every level** has $\mu_i \ge h(G)^2$.

Cheeger proof magically does this!