"Expander" Graphs: aside

$G = (V, E)$

Conductance: $\min_S |E(S, V-S)| / \min(|E(S)|, |E(V-S)|)$.
Edge Expansion: $\min_S E(S, V-S) / \min(|S|, |V-S|)$.
Vertex Expansion: $\min_{S, |S| \le n/2} |N(S)| / |S|$.
Sparsity(up to scaling): $\min_S |E(S, V-S)| / (|E(S)| \times |E(V-S)|)$

Complete Graph: (uniform degree)
  Sparsity: $\frac{n/2 \times n/2}{\binom{n}{2}} \approx 1/2$.

Hypercube:
  Best Sparsity: $1/d$.
    Dimension cut.
    (Vertex expansion: 1, each vertex has partner in other subcube..)
  Vertex Expansion: $\Theta(1/\sqrt{d})$. $\binom{d}{d/2} \propto \frac{2^d}{\sqrt{d}}$
    Majority cut.
    Sparsity: $\Theta(1/\sqrt{d})$ $d/2$ edges to vertices with more 1's.

Degree d expander graph.
  Sparsity: $\Omega(1)$. Vertex expansion: $\Omega(1)$. Small sets: $\Omega(1/d)$.
  Construction: randomized, or explicit e.g., group theory, zig-zag,...

Spielman-Teng $\to \cdots \to$ Kyung-Sachdeva

Solve $B^{-1}Ax = b$, where B is easy to invert.

Spielman-Teng:
  Form B by finding subclusters that are expanders,
    ultra-sparsify using low stretch trees, eliminate, sparsify, ...

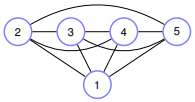
Kind of a big mess. Tons of ideas.

Gaussian Elimination

$Ax = b$.

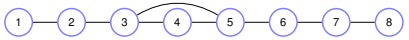
$x_1 = b_1 - a_1 \cdot x$
$a'_{2,1} = 0$
$\forall j \ge 1, a'_{2,j} = a_{2,j} - \frac{a_{2,1}}{a_{1,1}} a_{1,j} \quad b'_2 = b_2 - \frac{a_{2,1}}{a_{1,1}} b_1$

$\vdots$



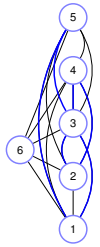
Elminate variable: graph view.
$\forall j \ge 1, a'_{2,j} = 0 - \frac{a_{2,1}}{a_{1,1}} a_{1,j}$

Expander: eliminate any half of vertices, (almost) get complete graph!
  Really complex.
Path:



Kyung-Sachdeva

Elimination/sparsify.
  Iteratively: $Ax_n = b$.
  Find $x_{n+1} = A^{-1}(b - Ax_n)$
    Would be done in one iteration!
    Circular!



Eliminate a vertex.
The neighbors would form a complete graph.
Implicitly Sparsify:
  Select edge with probability p, multiply by $1/p$.
    every edge unchanged in expectation.
    fewer edges though.
(Approximate Cholesky factorization.)

Didn't really solve $x_{n+1} = (A^{-1})(b - Ax_n)$.
Argued elimination process approximates eigenvalues of A^{-1}
Good condition number for $\tilde{A}^{-1}A$.
  Note: the $\tilde{A}^{-1}$ is fake notation representing elimination. No matrix $\tilde{A}$.

Algorithms.

Iterative good on fast communication systems: lots of short paths.

Elimination good on slow communication,
  not very connected systems.

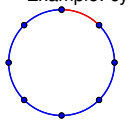
Nested Disection:
  Planar graphs (elimination) in $O(n^{3/2})$ time:
    complete graph only on separators.

Tradeoff in general:
  maybe combine iterative $1/\sqrt{n}$-expanders
    and eliminate between them (iteratively)?

Preconditioning:
  Solve $B^{-1}Ax = B^{-1}b$, where B is easy to invert.
  And $B^{-1}A$ has good condition number, i.e., λ_1/λ_n.
    if $B = A$, $B^{-1}A = I$, where condition number is 1.
  Vaidya: if B max weight spanning tree, condition number $O(m)$
    regardless of weights on edges!

Low stretch trees.

Another cool approach:
  Low stretch spanning tree.
    Spanning tree T with small $\ell_T(u, v)$ for average edge $e = (u, v)$.
    Example: expander. Any short tree. Diameter is $O(\log n)$.
    Example: cycle.



Distance 1 goes to $n-1$!

For cycle, remove a random edge get a tree.

  Average Stretch of edge: $\frac{n-1}{n} \times 1 + \frac{1}{n} \times (n-1) \approx 2$

In general: $\tilde{O}(m) = O(m \log n \log\log n)$.

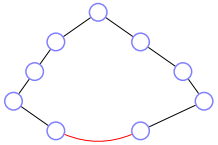
KOSZ

Laplacian systems $\equiv$ to routing electrical flow with sources and sinks.

Solve $\phi(u)$ and currents, $f(\cdot)$, on spanning tree.

$f(e) = (\phi(u) - \phi(v))/r(e) = \phi(u) - \phi(v) = \phi(u,v)$.
(Assume resistances $r(e) = 1$.)

Pick non-tree edge with probability $\propto \ell_T(e) + 1$



$f(e) \neq \phi(u,v)$

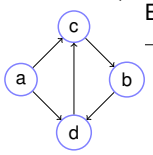
Change flow by $\frac{f(u,v)-\phi(u,v)}{\ell_T(e)+1}$.
Reduces $\sum_{e=(u,v)}(f(u,v) - \phi(u,v))^2$
Increases a bit on $\ell_T(e)$ (no violation) edges)
reduces a lot on e since large violation

Decrease on one edge dominate increase on many edges?
No violation on many, large violation on one. Quadratic.
Increase: $\ell \times (\varepsilon^2)$, decrease $1^2 - (1-\varepsilon)^2 = -2\varepsilon + \varepsilon^2$.
Change: $-2\varepsilon + (\ell+1)\varepsilon^2$. $\varepsilon = \frac{1}{\ell+1} \implies -\varepsilon$ decrease.

Some Matrices.

Given $G = (V, E)$, arbitrarily orient edges.

$$B_{v,e} = \begin{cases} -1 & e = (u,v) \\ 1 & e = (v,u) \\ 0 & \text{otherwise} \end{cases} \qquad L_{u,v} = \begin{cases} d & u = v \\ -1 & (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$



B

	a	b	c	d
(a,b)	1	-1	0	0
(a,c)	1	0	-1	0
(c,d)	0	0	1	-1
(d,b)	0	-1	0	1
(b,c)	0	1	-1	1

L

	a	b	c	d
a	2	-1	-1	0
b	-1	2	0	-1
c	-1	-1	3	-1
d	-1	-1	-1	3

Fun facts: $\mathbf{f} \in \Re^{|E|}$
$[B^T f]_u = \sum_{e=(u,v)} f_e - \sum_{e=(v,u)} f_e$
$B^T B = L$
$[Bx]_{e=(u,v)} = x_u - x_v$
$x^T L x = \sum_{e=(u,v)} (x_u - x_v)^2$

Analysis Idea

Total violation: $\sum_{e=(u,v)}(f(e) - \phi(u,v))^2$.

Edge w/stretch ℓ selected: reduce violation by $\approx \frac{1}{\ell}(f(e) - \phi(u,v))^2$.

Non-tree edge reduces $\propto \alpha(f(e) - \phi(u,v))^2$
tree edges increase $\propto \ell(\alpha^2(f(e) - \phi(u,v))^2$
$\alpha = \Theta(1/\ell) \implies$ reduction $\propto \frac{1}{\ell}(f(e) - \phi(u,v))^2$.

Selected $\propto \ell$.

$(f(e) - \phi(u,v))^2 \times \frac{1}{\ell} \times \ell$
$\implies$ expected reduction $\propto \sum_{e=(u,v)}(f(e) - \phi(u,v))^2$.

The constant of proportionality: $\sum_e(\ell_T(e)+1). = \tilde{O}(m)$
Probabilities: $\approx \frac{\ell_e+1}{\sum_e(\ell_T(e)+1)}$

$(1 - \tilde{\Omega}(1/m))$ expected reduction.

$\tilde{O}(m)$ iterations halves error.

Again: used exact solve on tree to speed up iterative method.
Shout out to "Algebra".

Update efficiently? Update on paths, path decomposition of trees.

Duality..

Given G, χ, $\chi \perp 1$

Minimize $|f|^2$ subject to $B^T f = \chi$.

Lagrangian: $L(\phi, f) = \sum_e f(e)^2 + 2\phi^T(\chi - B^T f)$

Lagrangian Dual: Find ϕ that maximizes $\min_f L(\phi, f)$.

Given ϕ, minimize $L(\phi, f)$? Calculus.

For $e = (u,v)$
$2f(e) + 2(\phi_v - \phi_u) = 0$ (Minimum when partial derivatives = 0.)

$\rightarrow f(e) = (\phi_u - \phi_v)$ Potential differences!!!

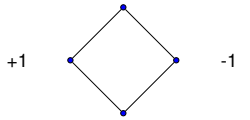
Matrix Form: $f = B\phi$ Again, flows should be potential differences.

Dual problem: Find ϕ that maximizes ...
$\max L(\phi, B\phi) = \phi^T B^T B\phi + 2\phi^T(\chi - B^T B\phi) \implies \max_\phi 2\phi^T\chi - \phi^T L\phi$
Note: want $\phi^T L\phi = \sum_e(\phi_u - \phi_v)^2$ to be small.
Minimize Squared Potential differences!

Electrical Flow and Laplacian Systems.

A graph $G = (V, E)$.
Circuit: nodes V, resistors E, value 1 (for today.)

Given $\chi : V \rightarrow \Re$
Find flow that routes χ and minimizes
$\sum_e f(e)^2$.



Claim: Minimizer is electrical flow.

Flow corresponds to flow induced by a set of potentials.

Why did we take dual?

Dual problem:
Find ϕ that maximizes ...
$\max_\phi 2\phi\chi - \phi L\phi$

Take the derivative:
$L\phi - \chi$
$L\phi = \chi$ at optimal point!

Optimal potential is solution to a Laplacian linear system.

Also useful for convergence.

Algorithm maintains feasible ϕ, f,

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi^T L\phi$

Duality gap is "distance" from optimal!

Algorithm: Work on flow and potentials.
To drive gap to 0.

Alg.

Given: χ, G

Take a spanning tree T of G. (Which tree?)

Route flow, f, to satisfy χ through T
Compute, ϕ, using tree ; $\phi_s = 0$, add f_e through T

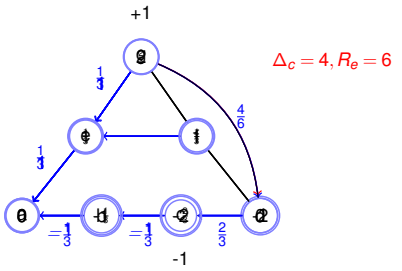
Repeat:
Choose non-tree edge $e = (u,v)$ (Which non-tree edge?)
$f(e) = (\phi_u - \phi_v)/(\ell_T(u,v)+1)$
($\ell_T(u,v)$ path length in T)
Route excess on path through tree.

Which Tree?
Claim: Linear time algorithm for T w/ stretch $O(m \log n \log\log n)$!
Stretch: $\sum_{e=(u,v)} \ell_T(u,v)$

Which non-tree edge?
Choose an edge w/prob. proportional to $\ell_T(e)$.

Finds $(1+\varepsilon)$ approximation in $O(m \log n \log\log n \log(\frac{n}{\varepsilon}))$! ! ! ! ! !

Animation



Energy reduction.

Given $T, e = (u,v)$, let $R_e = \ell_T(u,v)+1$.

Algorithm:

Repeatedly "Fix" edge $e = (u,v)$.
Route $-\delta = -\frac{\sum_{e' \in C_e} f(e')}{R_e}$ flow around cycle induced in T: C_e
(assume e' are oriented around cycle.)

Difference in energy from f and f'.
$\sum_{e' \in C_e} (f(e') - \delta)^2 - (f(e'))^2 = \sum_{e' \in C_e} -2f(e')\delta + \delta^2$
$= -(2\delta \sum_{e' \in C_e} f(e')) + R_E \delta^2$

Note: $\sum_{e' \in C_e} f(e') = R_e \delta$

$\rightarrow -\Delta_{C_e}^2 / R_e$ where $\Delta_{C_e} = \sum_{e' \in C_e} f(e')$.

Fix a part of the potential difference, Δ_{c_e} around cyle!!

$\rightarrow$ reduction of $\Delta_{C_e}^2 / R_e$ in energy!

Fix $1/R_e$ of a cycle violation!

Duality Gap?

Algorithm maintains feasible ϕ, f, ($B^T f = \chi$)

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi L \phi$

ϕ is tree induced voltages.

Total Duality Gap?

Gap: $|f|^2 - (2\phi^T \chi - \phi^T L \phi)$.
$= |f|^2 - 2\phi^T B^T f + \phi^T B^T B \phi$ where $B^T f = \chi$ and $L = B^T B$.
$= (f - B\phi)^T (f - B\phi)$.

Gap $= \sum_e (f(e) - \Delta_\phi(e))^2$ Difference between ϕ flow and f.

$\Delta_\phi(u,v) = \sum_{e \in P_{u,v}} -f(e)$. assume $f(e)$ is oriented around cycle.

For $e \in T$, $\Delta_\phi(e) = f(e)$. For $e \notin T$ $\Delta_{C_e} = f(e) + \sum_{e \in P_e} f(e)$

Duality Gap: $\sum_{e \notin T} (\Delta_{C_e}(f))^2$

Total distance from optimal is cycle violations!

Polishing off.

Claim: E[change in energy|Gap] $= \frac{Gap}{\tau}$
($\tau = \sum_e \ell_T(e)$ is stretch of E in T.)

Duality Gap: $\sum_{e \notin T} \Delta_{C_e}(f)^2$

Choose edge e reduce energy by $-\frac{\Delta_{C_e}^2}{R_e}$.

Choose edge with probability $\frac{R_e}{\tau}$.

Expected reduction $-\sum_e \frac{R_e}{\tau} \frac{\Delta_{C_e}^2}{R_e} = -\sum_e \frac{\Delta_{C_e}^2}{\tau}$ □

Duality Gap reduces by $(1 - 1/\tau)$ every iteration on expectation.

$O(\tau \log(n/\varepsilon))$ iterations gives $(1+\varepsilon)$ approximation.

$\tau = O(m \log n \log\log n)$ ...

$\tilde{O}(m)$ iterations

Iteration in $O(\log^2 n)$ time using balanced binary trees.

$\rightarrow \tilde{O}(m)$ time! ! ! ! ! !
Enough with the exclamations already !

Wrapup...

How to get low stretch tree?
Answer: Elkin-Spielman-Teng, ...,Abraham, Newman.
Open: Get $O(m \log n)$ stretch? Like for HSTs. (Later)
Ideas: Similar to HST construction, but much more subtle.

How to do update along cycle?
Answer: Data Structures.
Idea: Use a binary tree on paths.
Decompose tree into paths.

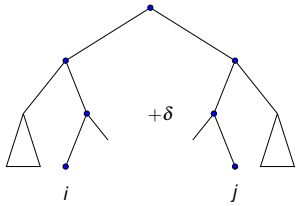
Interval Tree

Path: $G=(V,E)$, $V=\{1,\ldots,n\}$, $E=\{(i,i+1) : i \in \{1,n-1\}\}$

Init: $f(e)=0$, $\forall e \in G$.

Update(i,j,δ)
$\forall k \in \{i,\ldots,j-1\}$, $f(i,i+1)=f(i,i+1)+\delta$

Lookup(i,j)
Return $\sum_{k=i}^{j-1} f(k,k+1)$.



Update: find common ancestor.
For left path:
Right children, add δ to edge.
Reflect for right path.

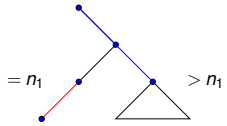
Lookup: Exercise.
Also, need number of descendants.

$O(\log n)$ update/lookup.

Path Decomposition of Tree

Given tree T.

Decomposition Procedure:
Longest path from root to leaf.
From root, go towards heavier side.
Remove. Recurse



Decomposition Property:
Where a root to leaf path in tree only sees $O(\log n)$ paths.

Proof Idea:
Every path change, doubles number of vertices.

$O(\log^2 n)$ update time for updating flow values on tree edges!

What's going on?

Geometric View.
Cycles are constraints.
Flow around cycle $= 0$.
Each cycle update is approximate projection into subspace defined by constraint.
The solution is intersection of all cycle constraints and flow conservation.

Algorithmic Power. Algebra: Solving exactly on tree.
Calculus: make a local move.

Better Algorithm:
Recursive algorithm give $O(m\sqrt{\log n})$ iterations to halve error.

Correspondence to Practice: Random sparsification of Cholesky factorization.

Laplacian Systems are quite general:
Climate, physics, SDD-matrices.