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$$M = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

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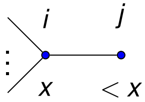

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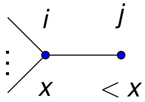

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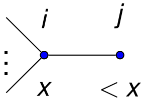

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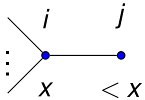

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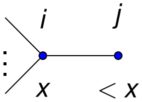

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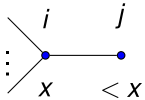

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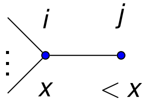

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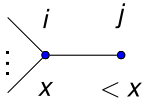

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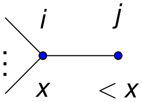

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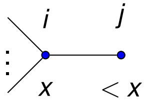

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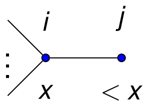

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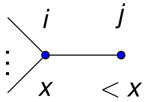

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Degree d, think of random ± 1 labels.

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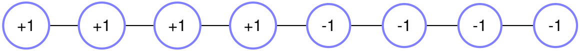

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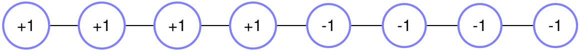

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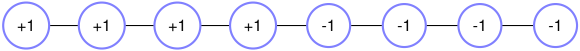

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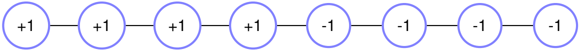

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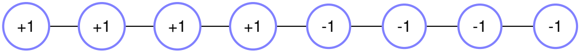

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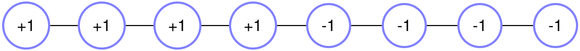

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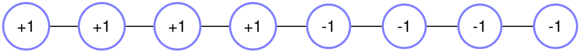

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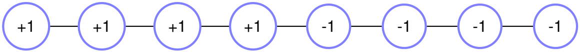

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$$x_i = \begin{cases} i - n/4 & \text{if } i \leq n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$

Hit with M.

$$(Mx)_i = \begin{cases} -n/4 + 1/2 & \text{if } i = 1, n \\ n/4 - 1 & \text{if } i = n/2 \\ x_i & \text{otherwise} \end{cases}$$

$\rightarrow x^T M x = x^T x(1 - O(\frac{1}{n^2})) \quad \rightarrow \lambda_2 \geq 1 - O(\frac{1}{n^2})$

$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$

$h(G) = \frac{2}{n} = \Theta(\sqrt{\mu})$

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Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

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$(Mx)_i$

Eigenvalues of cycle?

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$x_i = \cos\frac{2\pi k i}{n}$

$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right)$

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

$x_i = \cos \frac{2\pi k i}{n}$

$$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right) = 2\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi k i}{n}\right)$$

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Eigenvalue: $\propto \cos\frac{2\pi k}{n}$.

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Eigenvalues:
vibration modes of system.

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Eigenvalues:

vibration modes of system.

Fourier basis.

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Eigenvalues:
- vibration modes of system.
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Hypercube

$V = \{0, 1\}^d$

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$V = \{0, 1\}^d \quad (x, y) \in E$

Hypercube

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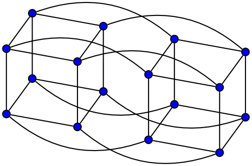

Good cuts?

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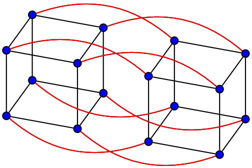

Good cuts? "Coordinate cut": d of them.

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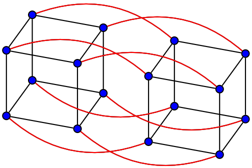

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Edge expansion:

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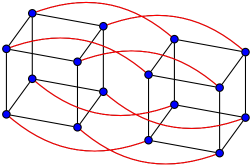

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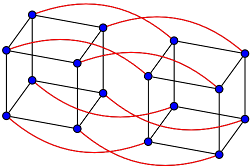

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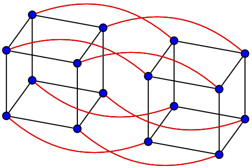

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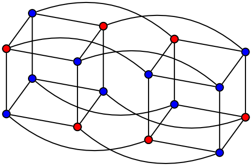

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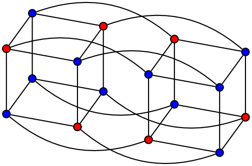

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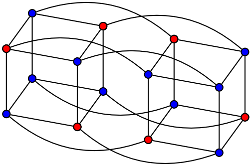

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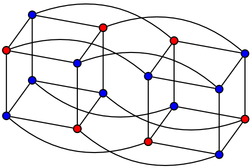

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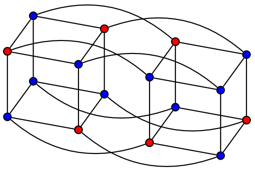

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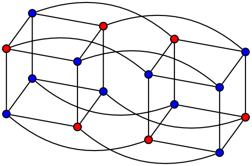

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Worse by a factor of $\sqrt{d}$

Eigenvalues of hypercube.

Anyone see any symmetry?

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Coordinate cuts. +1 on one side, -1 on other.

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Eigenvalue $1-2/d$.

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Eigenvalue $1-2/d$. d Eigenvectors.

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Next eigenvectors?

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Four subcubes: bipartite.

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Eigenvalue: $1-4/d$.

Eigenvalues of hypercube.

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Eigenvalue: $1-4/d$. $\binom{d}{2}$ eigenvectors.

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Eigenvalues: $1-2k/d$. $\binom{d}{k}$ eigenvectors.

Back to Cheeger.

Coordinate Cuts:

Back to Cheeger.

Coordinate Cuts:
Eigenvalue $1-2/d$.

Back to Cheeger.

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$\frac{\mu}{2}$

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Coordinate Cuts:
Eigenvalue $1-2/d$. d Eigenvectors.

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Eigenvalue $1-2/d$. d Eigenvectors.

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For hypercube: $h(G) = \frac{1}{d}$

Back to Cheeger.

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$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

For hypercube: $h(G) = \frac{1}{d}$ $\lambda_1 - \lambda_2 = 2/d$.
Left hand side is tight.

Back to Cheeger.

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Note: hamming weight vector also in first eigenspace.

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Lose “names” in hypercube, find coordinate cut?

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Find coordinate cut?

Back to Cheeger.

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For hypercube: $h(G) = \frac{1}{d}$ $\lambda_1 - \lambda_2 = 2/d$.
Left hand side is tight.

Note: hamming weight vector also in first eigenspace.

Lose "names" in hypercube, find coordinate cut?

Find coordinate cut?

Eigenvector v maps to line.

Back to Cheeger.

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Recall drunken sailor.