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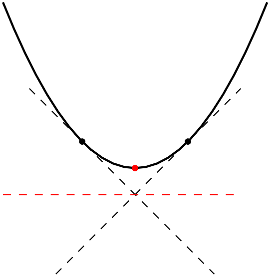
Translate using L: $f(x_t) - f(x^*) \leq \frac{L}{2}\|x_t - x^*\|^2$.

Mirror Descent



- Each point gives a linear lower bound.
- Average of the lower bounds becomes flatter.
- Add the point with current worst regret.
- Output average of queried points.
- $x = \alpha x_1 + (1 - \alpha) x_2$

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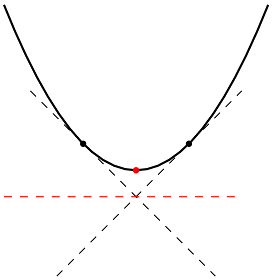

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- $x = \alpha x_1 + (1-\alpha) x_2$
  $\implies f(x) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$.

Analysis doesn't require L-Lipschitz.

Mirror Descent: Regret Minimization

- Average **Regret** with loss vector ξ_i's

$$R_k(u) = \frac{1}{k}\sum_{i=0}^{k-1}\langle \xi_i, z_i - u\rangle$$

Why care about average regret? Bounds gap to OPT:

With $\xi_i = \nabla f(z_i)$, $\bar{z} = \frac{1}{k}\sum_{i=0}^{k-1} z_i$,

$$f(\bar{z}) - f(u) \leq \frac{1}{k}\sum_{i=0}^{k-1} f(z_i) - f(u) \leq \frac{1}{k}\sum_{i=0}^{k-1}\langle \nabla f(z_i), z_i - u\rangle = R_k(u)$$

$$f(\bar{z}) - \mathrm{OPT} \leq \max_u R_k(u)$$

Mirror Descent: Regret Minimization

- Regularized average regret

$$\tilde{R}_k(u) = \frac{1}{\alpha k}\left(-w(u) + \alpha \sum_{i=0}^{k-1} \langle \xi_i, z_i - u \rangle\right)$$
$$= R_k(u) - \frac{w(u)}{\alpha k}$$

Distance Generating Function

$$w : \mathbb{R}^n \to \mathbb{R}$$

1-strongly convex for norm $\|\cdot\|$:

$$w(y) \geq w(x) + \langle \nabla w(x), y - x \rangle + \frac{1}{2}\|x - y\|^2$$

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For ℓ_2-norm, simply $w(x) = \frac{1}{2}\|x\|_2^2$.
(For distributions: $w(x) = -\sum_i x_i \log x_i$.)

Bregman divergence

$$V_x(y) = w(y) - \langle \nabla w(x), y - x \rangle - w(x) \geq \frac{1}{2}\|x - y\|^2$$

Standard three point property of Bregman divergence:

$$\forall x, y \geq 0 \quad \langle -\nabla V_x(y), y - u \rangle = V_x(u) - V_y(u) - V_x(y),$$

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Three point property $\leftrightarrow$ Law of cosines

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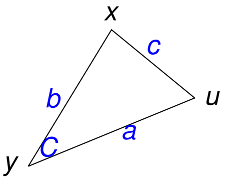

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

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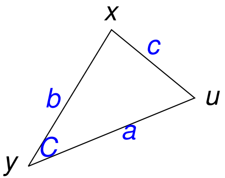

$$c^2 = a^2 + b^2 - 2ab\cos(C) \text{ or } 2ab\cos(C) = c^2 - a^2 - b^2$$

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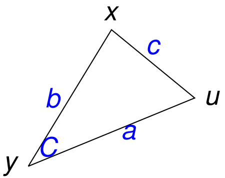

$$c^2 = a^2 + b^2 - 2ab\cos(C) \text{ or } 2ab\cos(C) = c^2 - a^2 - b^2$$

$$a^2 = V_y(u), \; b^2 = V_x(u), \quad c^2 = V_x(u), \; 2ab\cos(C) = -(x - y)\cdot(y - u)$$

Mirror Descent

$$z_{k+1} = \mathrm{Mirr}(z_k, \alpha\xi_k) = \underset{z\in Q}{\mathrm{argmin}}\{V_{z_k}(z) + \alpha\langle\xi_k, z - z_k\rangle\}$$

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Equivalent to regret minimization when $Q = \mathbb{R}^n$:

- Optimality condition of MD step:

$$\begin{aligned}\nabla V_{z_k}(z_{k+1}) &= -\alpha\xi_k \\ z_{k+1} - z_k &= -\alpha\xi_k \\ z_{k+1} &= z_0 - \sum_i \alpha\xi_i\end{aligned}$$

- Regret Minimization:

$$z_{k+1} = \underset{z}{\text{argmax}}\{-w(z) + \alpha\sum_{i=0}^{k}\langle\xi_i, z_i - z\rangle\}$$

  Optimality condition:

$$z_{k+1} = -\sum_i \alpha\xi_i$$

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Recall: $\nabla V_{z_k}(z_{k+1}) = -\alpha\xi_k \quad z_{k+1} - z_k = -\alpha\xi_k$

Lemma:

$$\begin{aligned}\alpha\langle\xi_k, z_k - u\rangle &\le \alpha\langle\xi_k, z_k - z_{k+1}\rangle + V_{z_k}(u) - V_{z_{k+1}}(u) - V_{z_k}(z_{k+1}) \\ &\le \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u) \qquad \forall u \in Q\end{aligned}$$

$$\begin{aligned}&= \alpha\langle\xi_k, z_k - z_{k+1}\rangle - \langle\nabla V_{z_k}(z_{k+1}), z_{k+1} - u\rangle \\ &= \alpha\langle\xi_k, z_k - z_{k+1}\rangle + V_{z_k}(u) - V_{z_{k+1}}(u) - V_{z_k}(z_{k+1})\end{aligned}$$

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Proof:

$$\alpha\langle \xi_k, z_k - u\rangle = \alpha\langle \xi_k, z_k - z_{k+1}\rangle + \alpha\langle \xi_k, z_{k+1} - u\rangle$$

□

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- $\alpha\langle \xi_k, z_k - u\rangle \le \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

  Telescoping T iterations, and **width** $\|\xi_k\|_*^2 \le \rho^2$

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$$2 \cdot \frac{\alpha^2\rho^2}{2} + V_{z_0}(u) - \cancel{V_{z_1}(u)} + \cancel{V_{z_1}(u)} - V_{z_2}(u)$$

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  $$3 \cdot \frac{\alpha^2\rho^2}{2} + V_{z_0}(u) - \cancel{V_{z_1}(u)} + \cancel{V_{z_1}(u)} - \cancel{V_{z_2}(u)} + \cancel{V_{z_2}(u)} - V_{z_3}(u) \ldots$$

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- Regret terms $\frac{\alpha^2}{2}\|\xi_k\|_*^2$ accumulate, bound step size α. Bregman divergence terms telescope.

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For $w(x) = \frac{1}{2}\|x - y\|^2$,
$z_k = z_0 - \alpha \|\psi_i\|^2$.
Bound "regret".

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$w(x)$ strongly convex function with $L = 1$

"Divergence" function $V_x(y) = w(y) - (w(x) + \nabla(w(x)) \cdot (y-x))$.

For sequence of "gradients": ψ_i.

For $w(x) = \frac{1}{2}\|x-y\|^2$,

$z_k = z_0 - \alpha\|\psi_i\|^2$.

Bound "regret".

$\sum_i^k \psi_i \cdot (z_i - u) \leq \alpha^2 \sum_i^k \|\psi_i\|^2 + V_{z_k}(u) - V_{z_0}(u)$.

Mirror Descent.

Convex, continuous function: $f(x)$.

$f(y) \geq f(x) + \nabla f(x)(y-x)$.

No lipshitz condition necessary.

Gradient need not be continuous: absolute value.

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Loss is compare to linear lower bound on function value at u.

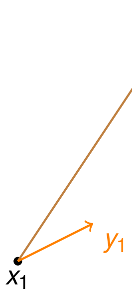
Linear Coupling

Intuition: If $\|\nabla f(x_k)\|_*^2$ large

- GD can make large primal progress $\frac{1}{2L}\|\nabla f(x_k)\|_*^2$
- MD suffers large regret $\frac{\alpha^2}{2}\|\nabla f(x)\|_*^2$
- Use primal progress to cover regret.
- Regret terms no longer accumulates,
  telescope as the primal progress.

Linear Coupling

- $x_0 = y_0 = z_0$.
- Coupling: $x_{k+1} = \tau z_k + (1 - \tau) y_k$.
- MD: $z_{k+1} = \text{Mirr}(z_k, \alpha \nabla f(x_{k+1}))$
- GD: $y_{k+1} = \text{Grad}(x_{k+1})$.



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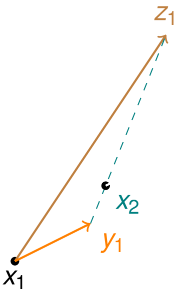

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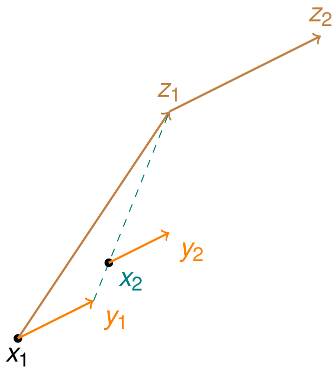

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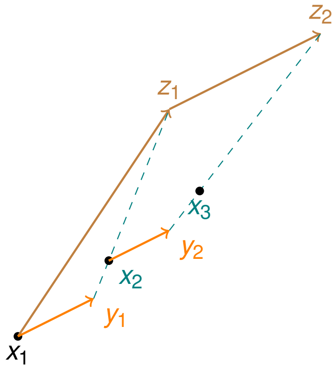

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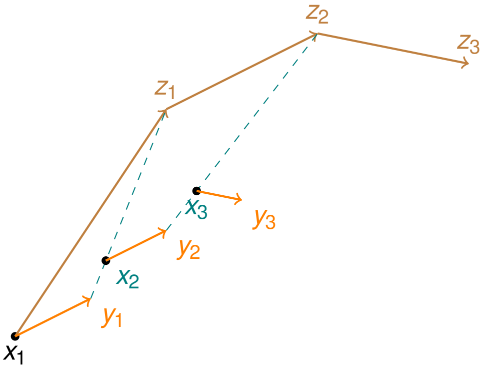
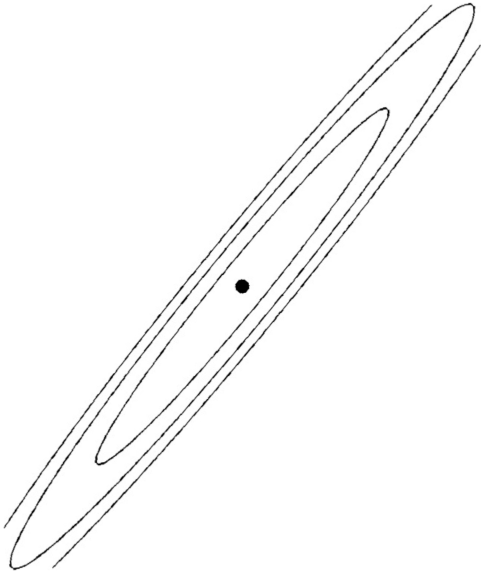


Linear Coupling

Momentum View:



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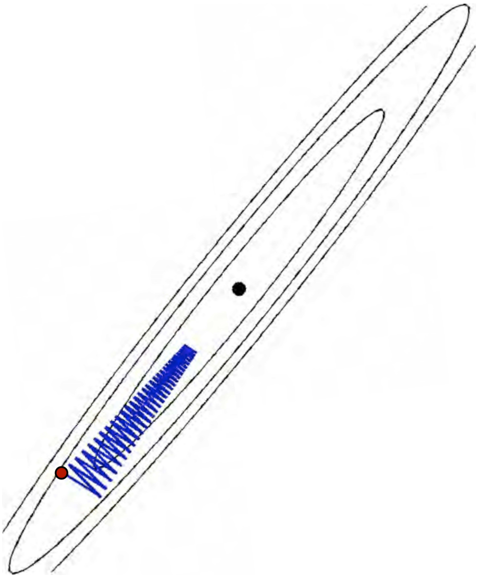

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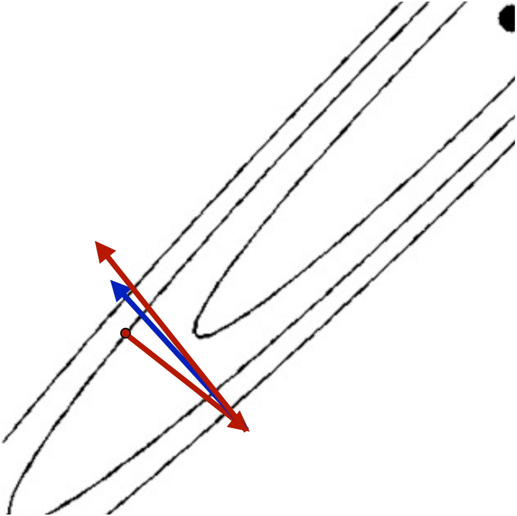

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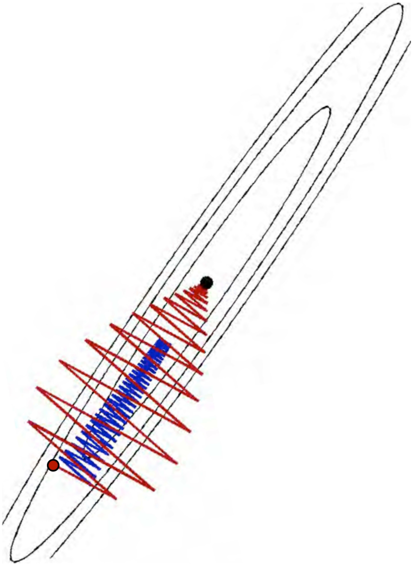
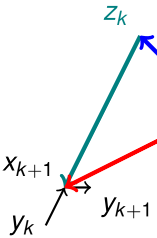
Momentum View:



Bound $\alpha(f(x_{k+1}) - f(u)) \leq \alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle$



$$\begin{aligned}&\alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle\\ =&\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle\\ &+\alpha\langle \nabla f(x_{k+1}), x_{k+1} - z_k\rangle\end{aligned}$$

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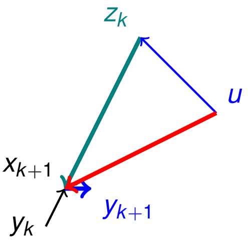

$$
\begin{aligned}
&\alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle \\
=&\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\
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\leq&\alpha^2 L(f(x_{k+1}) - f(y_{k+1})) \\
&+ V_{z_k}(u) - V_{z_{k+1}}(u)
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MD: $\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \leq \frac{\alpha^2}{2}\|\nabla f(x_{k+1})\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

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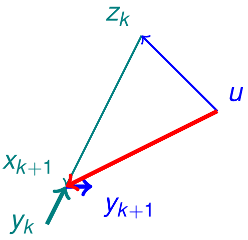

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Coupling:

$x_{k+1} = \tau z_k + (1-\tau)y_k \quad \rightarrow \quad \tau(x_{k+1} - z_k) = (1-\tau)(y_k - x_{k+1})$

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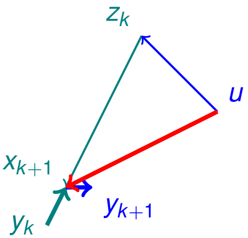

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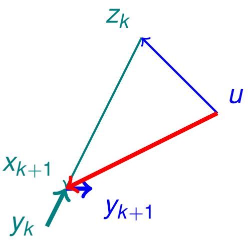

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Let $\alpha^2 L = \dfrac{1-\tau}{\tau}\alpha$

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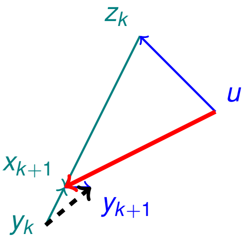

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Both components telescope!

Linear Coupling

- Summing over $0, \ldots, T-1$, with $\bar{x} = \frac{1}{T}\sum_i x_i$

$$f(\bar{x}) - f(u) \leq \frac{\alpha L}{T}(f(y_0) - f(y_T)) + \frac{V_{z_0}(u)}{\alpha T}$$

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- If $f(y_0) - \text{OPT} \leq d$, diameter $V_{z_0}(u) \leq \Theta$, $\alpha = \sqrt{\frac{\Theta}{Ld}}, T = 4\sqrt{\frac{L\Theta}{d}}$

$$f(\bar{x}) - f(u) \leq \frac{\alpha L d + \Theta/\alpha}{T} \leq \frac{d}{2}$$

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$$f(\bar{x}) - f(u) \le \frac{\alpha L d + \Theta/\alpha}{T} \le \frac{d}{2}$$

- In $T = 4\sqrt{\frac{L\Theta}{d}}$ iterations,

$$f(x_0) - \mathrm{OPT} \le d \quad \rightarrow \quad f(\bar{x}) - \mathrm{OPT} \le \frac{d}{2}$$

To get ε-approximation:

$$T = O(\sqrt{\frac{L\Theta}{\varepsilon}} + \sqrt{\frac{L\Theta}{2\varepsilon}} + \ldots) = O(\sqrt{\frac{L\Theta}{\varepsilon}})$$

Linear Coupling

- With $\alpha_k = \frac{k+1}{2L}$, can remove phases, and have $f(y_T) - f(u) \leq \varepsilon$ after $T = O(\sqrt{\frac{L\Theta}{\varepsilon}})$ iterations.
  Almost the same as Nesterov's.
- GD: $O(\frac{LR^2}{2})$ v.s. MD: $O(\frac{\rho^2\Theta}{\varepsilon^2})$ v.s. AGD: $O(\sqrt{\frac{L\Theta}{\varepsilon}})$

Spectral Theorem.

For real symmetric matrix, A, there exists a set of unit vectors $u_1, \ldots, u_n$, and real numbers λ_i such $v_i \perp v_j$ and $Au_i = \lambda_i u_i$.

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Fix λ.

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Minimizer is “eigenvector”.

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A has eigenpairs: $(u_1, \lambda_1)...(u_n, \lambda_n)$.

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