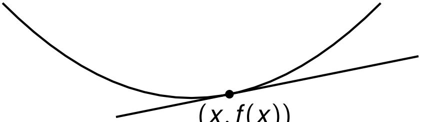
Convex optimization

Slides: Thanks to Di Wang.

$$\min_{x \in Q} f(x)$$

$$f(x) - f(y) \leq \langle \nabla f(x), x - y \rangle$$

Q: feasible space, convex.



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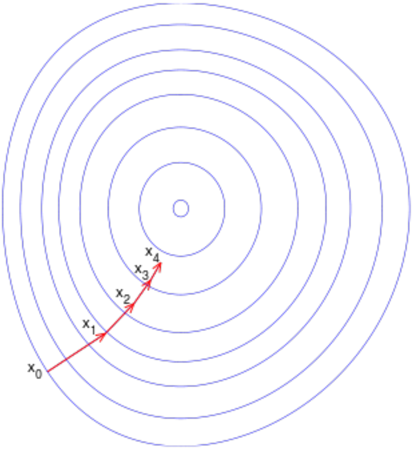
Q: feasible space, convex.

First-order Iterative Methods

- Query $x \in Q$, update using $\nabla f(x)$
- Low per-iteration cost, $\mathrm{poly}(\frac{1}{\varepsilon})$ convergence.
- Methods of choice in large-scale regime.

Gradient Descent



- Moves in down-hill direction.
- Improve objective function value each iteration.
- Output final point.

Gradient Descent

L-Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x - y\| \quad \forall x, y \in Q$$

- Global linear lower bound and quadratic upper bound:

$$\forall y \quad f(x) + \langle \nabla f(x), y - x \rangle \leq f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2$$

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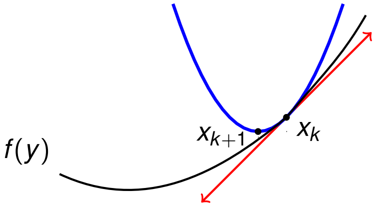

$f(x)$ rises faster when going up,
and falls slower when going down.

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- Minimize using quadratic bound

$$x_{k+1} = \mathrm{Grad}(x_k) = \operatorname*{argmin}_{x \in Q} \{ \langle \nabla f(x_k), x - x_k \rangle + \frac{L}{2}\|x - x_k\|^2 \}$$

If $Q = \mathbb{R}^n$ and ℓ_2-norm, $x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k)$.

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- **Primal progress**: Av. $\nabla f(x') \geq \frac{\nabla f(x)}{2}$ for $x' = \alpha x_k + (1 - \alpha) x_{k+1}$

$$f(x_k) - f(x_{k+1}) \geq \frac{1}{2L}\|\nabla f(x_k)\|_*^2$$

Gradient Descent: one dimensional intuition.

Convexity:
$f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$.

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In one dimension: $\nabla f(x) = g$.

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Gap: gR.

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Idea: Gap/(progress/step) $\implies$ roughly $2LR/g$ steps.

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While gap $f(x) - f(x^*) \geq \varepsilon$ we have $g \geq \varepsilon/R$.

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While gap $f(x) - f(x^*) \geq \varepsilon$ we have $g \geq \varepsilon/R$.

$\implies O(LR^2/\varepsilon)$ steps reduce gap by $1/2$.

Gradient Descent: convergence in ℓ_2

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$$\leq \frac{L}{2}\left(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$$

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$$\leq \frac{L}{2}\left(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2\right)$$
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$$\sum_k^T f(x_k) - f(x^*) \leq \sum_k^T \frac{L}{2}\left(\|x_{k-1} - x^*\|_2^2 - \|x_k - x^*\|_2^2\right)$$
$$\leq \frac{L}{2}\left(\|x_0 - x^*\|_2^2 - \|x_T - x^*\|_2^2\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2$$

Gradient Descent: convergence in ℓ_2

Convexity:

$$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x). \implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$$

L-Lipschitz, $R = \|x_0 - x^*\|$:

$$x^+ = x - \frac{1}{L}\nabla f(x) \qquad f(x) - f(x^+) \geq \frac{1}{2L}\|\nabla f(x)\|_*^2$$

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$f(x_k)$ is decreasing, we have $f(x_T) \leq \frac{1}{T}\sum_k f(x_k)$.

Gradient Descent: convergence in ℓ_2

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$\implies f(x_T) - f(x^*) \leq \frac{LR^2}{2T}$ where $R = \|x_0 - x^*\|$.

Gradient Descent: convergence in ℓ_2

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$f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x)$. $\implies f(x) \leq f(x^*) + \nabla f(x)^T(x - x^*)$

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$\leq \frac{L}{2}(\|x - x^*\|_2^2 - \|(x - x^*) - \frac{1}{L}\nabla f(x)\|_2^2)$

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$\implies f(x_T) - f(x^*) \leq \frac{LR^2}{2T}$ where $R = \|x_0 - x^*\|$.

Also: $T = O(LR^2/\varepsilon)$ iterations for $f(x_T) - f(x^*) \leq \varepsilon$.

Gradient Descent

Primal progress

$$f(x_k) - f(x_{k+1}) \geq \frac{1}{2L} \|\nabla f(x)\|_*^2$$

Convergence

L-Lipschitz, $R = \max_{x: f(x) \leq f(x_0)} \|x - x^*\|$:

$$f(x_T) - f(x^*) \leq O(\frac{LR^2}{T})$$

To get ε-approximation, need

$$T = O(\frac{LR^2}{\varepsilon})$$

Relationship?

What is relationship to move closer to feasible?

Relationship?

What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.

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If wrong side of hyperplane by at least something.
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$\nabla f(x)$

Relationship?

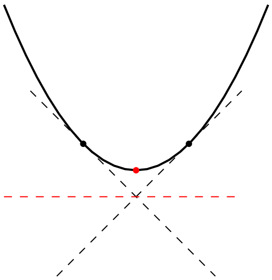
What is relationship to move closer to feasible?

If wrong side of hyperplane by at least something.
Move to other side.

What is the "hyperplane" here?

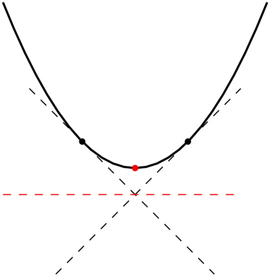
$\nabla f(x)$ Maybe.

Mirror Descent



- Each point gives a linear lower bound.
- Average of the lower bounds becomes flatter.
- Add the point with current worst regret.
- Output average of queried points.
- $x = \alpha x_1 + (1 - \alpha) x_2$

Mirror Descent



- Each point gives a linear lower bound.
- Average of the lower bounds becomes flatter.
- Add the point with current worst regret.
- Output average of queried points.
- $x = \alpha x_1 + (1-\alpha)x_2$
  $\implies f(x) \le \alpha f(x_1) + (1-\alpha)f(x_2)$.

Analysis doesn't require L-Lipschitz.

Mirror Descent: Regret Minimization

- Average **Regret** with loss vector ξ_i's

$$R_k(u) = \frac{1}{k}\sum_{i=0}^{k-1}\langle \xi_i, z_i - u\rangle$$

Why care about average regret? Bounds gap to OPT:

With $\xi_i = \nabla f(z_i)$, $\bar{z} = \frac{1}{k}\sum_{i=0}^{k-1} z_i$,

$$f(\bar{z}) - f(u) \le \frac{1}{k}\sum_{i=0}^{k-1} f(z_i) - f(u) \le \frac{1}{k}\sum_{i=0}^{k-1}\langle \nabla f(z_i), z_i - u\rangle = R_k(u)$$

$$f(\bar{z}) - \mathrm{OPT} \le \max_u R_k(u)$$

Mirror Descent: Regret Minimization

- Regularized average regret

$$\tilde{R}_k(u) = \frac{1}{\alpha k}(-w(u) + \alpha \sum_{i=0}^{k-1} \langle \xi_i, z_i - u \rangle)$$
$$= R_k(u) - \frac{w(u)}{\alpha k}$$

Distance Generating Function

$$w : \mathbb{R}^n \to \mathbb{R}$$

1-strongly convex for norm $\|\cdot\|$:

$$w(y) \geq w(x) + \langle \nabla w(x), y - x \rangle + \frac{1}{2}\|x - y\|^2$$

Mirror Descent: Regret Minimization

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For ℓ_2-norm, simply $w(x) = \frac{1}{2}\|x\|_2^2$.
(For distributions: $w(x) = -\sum_i x_i \log x_i$.)

Bregman divergence

$$V_x(y) = w(y) - \langle \nabla w(x), y - x \rangle - w(x) \geq \frac{1}{2}\|x - y\|^2$$

Standard three point property of Bregman divergence:

$$\forall x, y \geq 0 \quad \langle -\nabla V_x(y), y - u \rangle = V_x(u) - V_y(u) - V_x(y),$$

Bregman divergence

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For ℓ_2-norm, $V_x(y) = \frac{1}{2}\|x - y\|_2^2$, $\nabla V_x(y) = (x - y)$

Bregman divergence

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Three point property $\leftrightarrow$ Law of cosines

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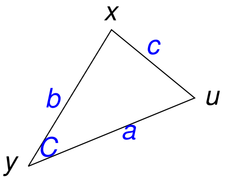
Three point property $\leftrightarrow$ Law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Bregman divergence

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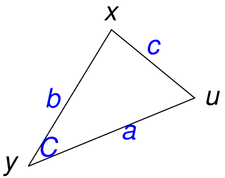

$$c^2 = a^2 + b^2 - 2ab\cos(C) \text{ or } 2ab\cos(C) = c^2 - a^2 - b^2$$

Bregman divergence

$$V_x(y) = w(y) - \langle \nabla w(x), y - x \rangle - w(x) \geq \frac{1}{2}\|x - y\|^2$$

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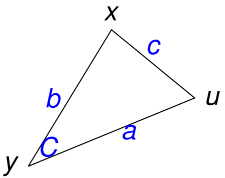

$$c^2 = a^2 + b^2 - 2ab\cos(C) \text{ or } 2ab\cos(C) = c^2 - a^2 - b^2$$

$$a^2 = V_y(u), \; b^2 = V_x(u), \quad c^2 = V_x(u), \; 2ab\cos(C) = -(x - y) \cdot (y - u)$$

Mirror Descent

$$z_{k+1} = \mathrm{Mirr}(z_k, \alpha\xi_k) = \operatorname*{argmin}_{z \in Q}\{V_{z_k}(z) + \alpha\langle \xi_k, z - z_k\rangle\}$$

Mirror Descent

$$z_{k+1} = \mathrm{Mirr}(z_k, \alpha\xi_k) = \operatorname*{argmin}_{z \in Q} \{ V_{z_k}(z) + \alpha \langle \xi_k, z - z_k \rangle \}$$

Equivalent to regret minimization when $Q = \mathbb{R}^n$:

- Optimality condition of MD step:

$$\begin{aligned} \nabla V_{z_k}(z_{k+1}) &= -\alpha\xi_k \\ z_{k+1} - z_k &= -\alpha\xi_k \\ z_{k+1} &= z_0 - \sum_i \alpha\xi_i \end{aligned}$$

- Regret Minimization:

$$z_{k+1} = \operatorname*{argmax}_{z} \{ -w(z) + \alpha \sum_{i=0}^{k} \langle \xi_i, z_i - z \rangle \}$$

Optimality condition:

$$z_{k+1} = -\sum_i \alpha\xi_i$$

Mirror Descent

$$z_{k+1} = \mathrm{Mirr}(z_k, \alpha\xi_k) = \operatorname*{argmin}_{z \in Q}\{V_{z_k}(z) + \alpha\langle \xi_k, z - z_k\rangle\}$$

Recall: $\nabla V_{z_k}(z_{k+1}) = -\alpha\xi_k \quad z_{k+1} - z_k = -\alpha\xi_k$

Lemma:

$$\begin{aligned}\alpha\langle \xi_k, z_k - u\rangle &\leq \alpha\langle \xi_k, z_k - z_{k+1}\rangle + V_{z_k}(u) - V_{z_{k+1}}(u) - V_{z_k}(z_{k+1}) \\ &\leq \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u) \qquad \forall u \in Q\end{aligned}$$

$$\begin{aligned}&= \alpha\langle \xi_k, z_k - z_{k+1}\rangle - \langle \nabla V_{z_k}(z_{k+1}), z_{k+1} - u\rangle \\ &= \alpha\langle \xi_k, z_k - z_{k+1}\rangle + V_{z_k}(u) - V_{z_{k+1}}(u) - V_{z_k}(z_{k+1})\end{aligned}$$

Mirror Descent

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Proof:

$$\alpha\langle \xi_k, z_k - u\rangle = \alpha\langle \xi_k, z_k - z_{k+1}\rangle + \alpha\langle \xi_k, z_{k+1} - u\rangle$$

□

Mirror Descent

$$z_{k+1} = \mathrm{Mirr}(z_k, \alpha\xi_k) = \operatorname*{argmin}_{z\in Q}\{V_{z_k}(z) + \alpha\langle \xi_k, z - z_k\rangle\}$$

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Proof:

$$\begin{aligned}\alpha\langle \xi_k, z_k - u\rangle &= \alpha\langle \xi_k, z_k - z_{k+1}\rangle + \alpha\langle \xi_k, z_{k+1} - u\rangle\\ &= \alpha\langle \xi_k, z_k - z_{k+1}\rangle - \langle \nabla V_{z_k}(z_{k+1}), z_{k+1} - u\rangle\\ &= \alpha\langle \xi_k, z_k - z_{k+1}\rangle + V_{z_k}(u) - V_{z_{k+1}}(u) - V_{z_k}(z_{k+1})\\ &\le \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)\end{aligned}$$

□

Mirror Descent

- $\alpha\langle \xi_k, z_k - u\rangle \leq \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

  Telescoping T iterations, and **width** $\|\xi_k\|_*^2 \leq \rho^2$

Mirror Descent

- $\alpha\langle\xi_k, z_k - u\rangle \leq \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

  Telescoping T iterations, and **width** $\|\xi_k\|_*^2 \leq \rho^2$

$$1 \cdot \frac{\alpha^2\rho^2}{2} + V_{z_0}(u) - V_{z_1}(u)$$

Mirror Descent

- $\alpha\langle \xi_k, z_k - u\rangle \leq \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

  Telescoping T iterations, and **width** $\|\xi_k\|_*^2 \leq \rho^2$

$$2 \cdot \frac{\alpha^2\rho^2}{2} + V_{z_0}(u) - \cancel{V_{z_1}(u)} + \cancel{V_{z_1}(u)} - V_{z_2}(u)$$

Mirror Descent

- $\alpha\langle \xi_k, z_k - u\rangle \le \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

  Telescoping T iterations, and **width** $\|\xi_k\|_*^2 \le \rho^2$

  $$3 \cdot \frac{\alpha^2\rho^2}{2} + V_{z_0}(u) - \cancel{V_{z_1}(u)} + \cancel{V_{z_1}(u)} - \cancel{V_{z_2}(u)} + \cancel{V_{z_2}(u)} - V_{z_3}(u) \ldots$$

Mirror Descent

- $\alpha\langle \xi_k, z_k - u\rangle \le \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

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$$\alpha \sum_{i=0}^{T-1} \langle \xi_i, z_i - u\rangle \le \frac{\alpha^2 \rho^2 T}{2} + V_{z_0}(u) - V_{z_T}(u)$$

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- $\alpha = \frac{\varepsilon}{\rho^2}$, **diameter** $V_{z_0}(u) \le \Theta$, in $T = \frac{2\rho^2\Theta}{\varepsilon^2}$ iterations

  $$\forall u, f(\bar{z}) - f(u) \le \frac{\alpha\rho^2}{2} + \frac{V_{z_0}(u)}{\alpha T} \le \varepsilon$$

Mirror Descent

- $\alpha\langle \xi_k, z_k - u\rangle \leq \frac{\alpha^2}{2}\|\xi_k\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

  Telescoping T iterations, and **width** $\|\xi_k\|_*^2 \leq \rho^2$

$$\alpha \sum_{i=0}^{T-1} \langle \xi_i, z_i - u\rangle \leq \frac{\alpha^2 \rho^2 T}{2} + V_{z_0}(u) - V_{z_T}(u)$$

- $\alpha = \frac{\varepsilon}{\rho^2}$, **diameter** $V_{z_0}(u) \leq \Theta$, in $T = \frac{2\rho^2\Theta}{\varepsilon^2}$ iterations

$$\forall u, f(\bar{z}) - f(u) \leq \frac{\alpha\rho^2}{2} + \frac{V_{z_0}(u)}{\alpha T} \leq \varepsilon$$

- Regret terms $\frac{\alpha^2}{2}\|\xi_k\|_*^2$ accumulate, bound step size α.
  Bregman divergence terms telescope.

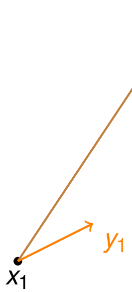
Linear Coupling

Intuition: If $\|\nabla f(x_k)\|_*^2$ large

- GD can make large primal progress $\frac{1}{2L}\|\nabla f(x_k)\|_*^2$
- MD suffers large regret $\frac{\alpha^2}{2}\|\nabla f(x)\|_*^2$
- Use primal progress to cover regret.
- Regret terms no longer accumulates,
  telescope as the primal progress.

Linear Coupling

- $x_0 = y_0 = z_0$.
- Coupling: $x_{k+1} = \tau z_k + (1-\tau) y_k$.
- MD: $z_{k+1} = \mathrm{Mirr}(z_k, \alpha \nabla f(x_{k+1}))$
- GD: $y_{k+1} = \mathrm{Grad}(x_{k+1})$.



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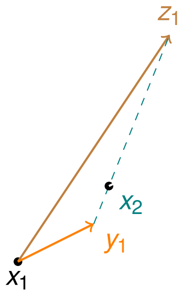

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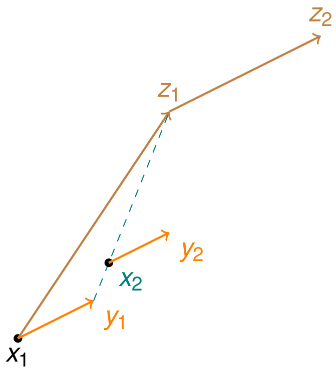

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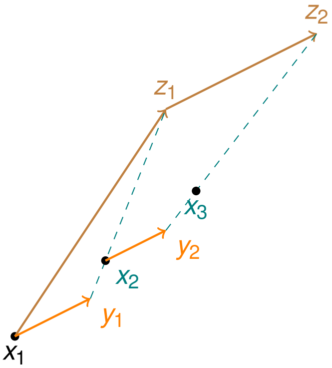

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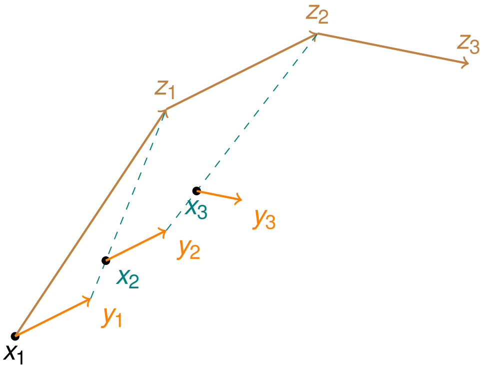
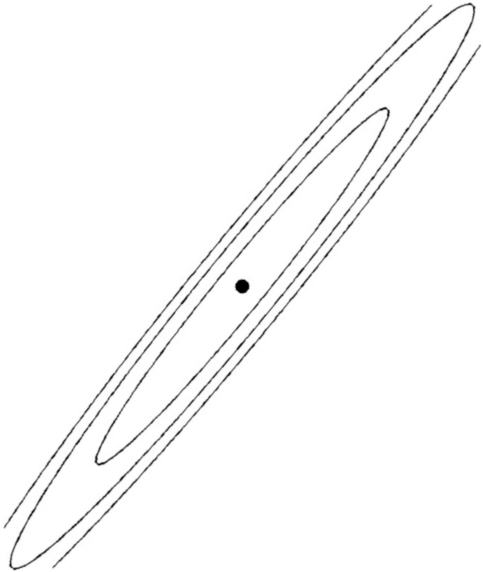


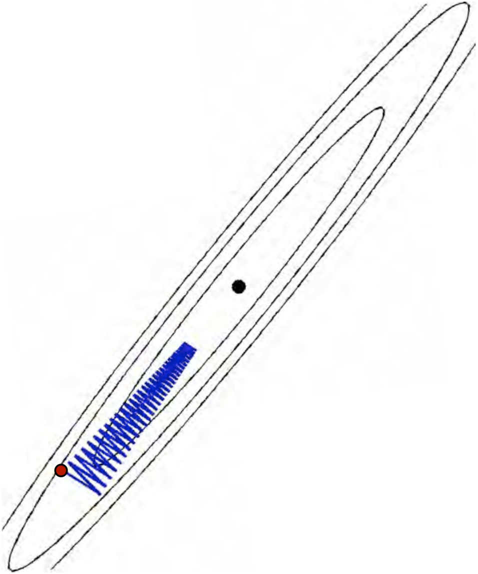
Linear Coupling

Momentum View:



Linear Coupling

Momentum View:



Linear Coupling

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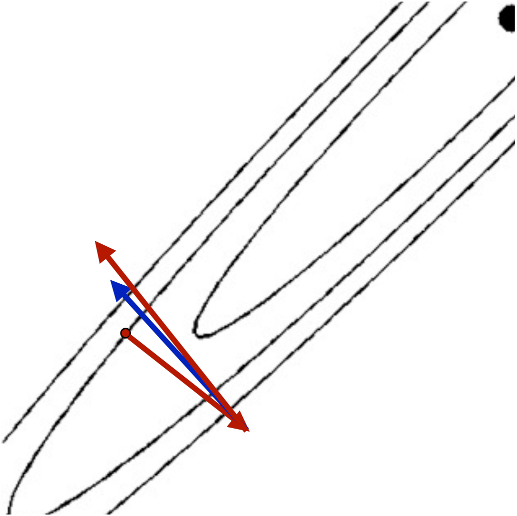
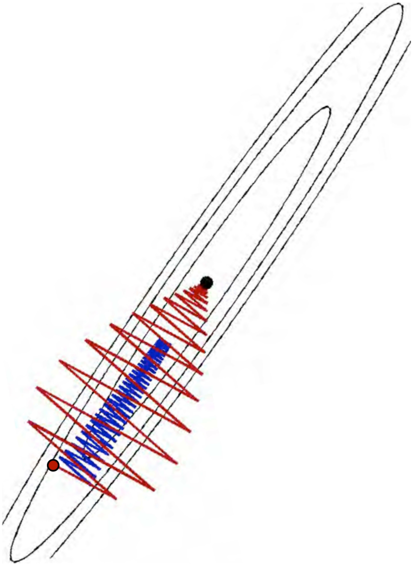
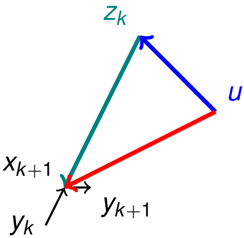


Linear Coupling

Momentum View:

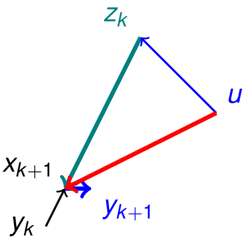


Bound $\alpha(f(x_{k+1}) - f(u)) \leq \alpha\langle\nabla f(x_{k+1}), x_{k+1} - u\rangle$



$$\begin{aligned} &\alpha\langle\nabla f(x_{k+1}), x_{k+1} - u\rangle \\ =&\alpha\langle\nabla f(x_{k+1}), z_k - u\rangle \\ &+ \alpha\langle\nabla f(x_{k+1}), x_{k+1} - z_k\rangle \end{aligned}$$

Bound $\alpha(f(x_{k+1}) - f(u)) \le \alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle$

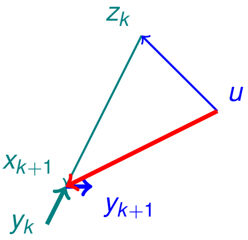


$$\begin{aligned} &\alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle \\ =&\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\ &+ \alpha\langle \nabla f(x_{k+1}), x_{k+1} - z_k\rangle \\ \le&\alpha^2 L(f(x_{k+1}) - f(y_{k+1})) \\ &+ V_{z_k}(u) - V_{z_{k+1}}(u) \end{aligned}$$

MD: $\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \le \frac{\alpha^2}{2}\|\nabla f(x_{k+1})\|_*^2 + V_{z_k}(u) - V_{z_{k+1}}(u)$

GD: $f(x_{k+1}) - f(y_{k+1}) \ge \frac{1}{2L}\|\nabla f(x_{k+1})\|_*^2$

Bound $\alpha(f(x_{k+1}) - f(u)) \leq \alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle$

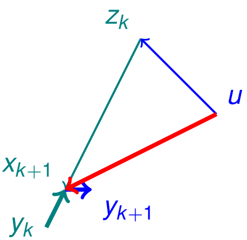


$$
\begin{aligned}
&\alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle \\
=&\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\
&+ \alpha\langle \nabla f(x_{k+1}), x_{k+1} - z_k\rangle \\
\leq&\alpha^2 L(f(x_{k+1}) - f(y_{k+1})) \\
&+ V_{z_k}(u) - V_{z_{k+1}}(u) \\
&+ \alpha\langle \nabla f(x_{k+1}), \frac{1-\tau}{\tau}(y_k - x_{k+1})\rangle
\end{aligned}
$$

Coupling:

$x_{k+1} = \tau z_k + (1-\tau) y_k \quad \rightarrow \quad \tau(x_{k+1} - z_k) = (1-\tau)(y_k - x_{k+1})$

Bound $\alpha(f(x_{k+1}) - f(u)) \le \alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle$



$$
\begin{aligned}
&\alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle \\
=&\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\
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&+ V_{z_k}(u) - V_{z_{k+1}}(u) \\
&+ \frac{1-\tau}{\tau}\alpha(f(y_k) - f(x_{k+1}))
\end{aligned}
$$

$$
\begin{aligned}
\alpha\langle \nabla f(x_{k+1}), \tfrac{1-\tau}{\tau}(y_k - x_{k+1})\rangle &= \alpha\tfrac{1-\tau}{\tau}\langle \nabla f(x_{k+1}), y_k - x_{k+1}\rangle) \\
&\le \tfrac{1-\tau}{\tau}\alpha(f(y_k) - f(x_{k+1}))
\end{aligned}
$$

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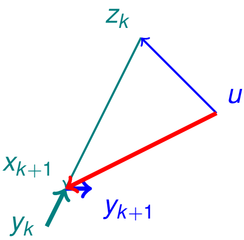

$$
\begin{aligned}
& \alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle \\
= & \alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\
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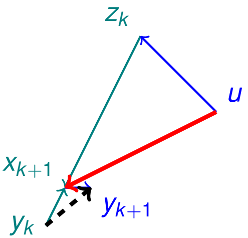
\end{aligned}
$$

Let $\alpha^2 L = \dfrac{1-\tau}{\tau}\alpha$

Bound $\alpha(f(x_{k+1}) - f(u)) \leq \alpha\langle\nabla f(x_{k+1}), x_{k+1} - u\rangle$



$$
\begin{aligned}
&\alpha\langle\nabla f(x_{k+1}), x_{k+1} - u\rangle \\
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&+ \alpha\langle\nabla f(x_{k+1}), x_{k+1} - z_k\rangle \\
\leq&\alpha^2 L(\cancel{f(x_{k+1})} - f(y_{k+1})) \\
&+ V_{z_k}(u) - V_{z_{k+1}}(u) \\
&+ \alpha^2 L(f(y_k) - \cancel{f(x_{k+1})})
\end{aligned}
$$

Let $\alpha^2 L = \frac{1-\tau}{\tau}\alpha$

Both components telescope!

Linear Coupling

- Summing over $0, \ldots, T-1$, with $\bar{x} = \frac{1}{T}\sum_i x_i$

$$f(\bar{x}) - f(u) \leq \frac{\alpha L}{T}(f(y_0) - f(y_T)) + \frac{V_{z_0}(u)}{\alpha T}$$

Linear Coupling

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$$f(\bar{x}) - f(u) \leq \frac{\alpha L}{T}(f(y_0) - f(y_T)) + \frac{V_{z_0}(u)}{\alpha T}$$

- If $f(y_0) - \mathrm{OPT} \leq d$, diameter $V_{z_0}(u) \leq \Theta$,
  $\alpha = \sqrt{\frac{\Theta}{Ld}}, T = 4\sqrt{\frac{L\Theta}{d}}$

$$f(\bar{x}) - f(u) \leq \frac{\alpha L d + \Theta/\alpha}{T} \leq \frac{d}{2}$$

Linear Coupling

- Summing over $0, \ldots, T-1$, with $\bar{x} = \frac{1}{T}\sum_i x_i$

$$f(\bar{x}) - f(u) \leq \frac{\alpha L}{T}(f(y_0) - f(y_T)) + \frac{V_{z_0}(u)}{\alpha T}$$

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$\alpha = \sqrt{\frac{\Theta}{Ld}}, T = 4\sqrt{\frac{L\Theta}{d}}$

$$f(\bar{x}) - f(u) \leq \frac{\alpha L d + \Theta/\alpha}{T} \leq \frac{d}{2}$$

- In $T = 4\sqrt{\frac{L\Theta}{d}}$ iterations,

$$f(x_0) - \mathrm{OPT} \leq d \quad \rightarrow \quad f(\bar{x}) - \mathrm{OPT} \leq \frac{d}{2}$$

To get ε-approximation:

$$T = O(\sqrt{\frac{L\Theta}{\varepsilon}} + \sqrt{\frac{L\Theta}{2\varepsilon}} + \ldots) = O(\sqrt{\frac{L\Theta}{\varepsilon}})$$

Linear Coupling

- With $\alpha_k = \frac{k+1}{2L}$, can remove phases, and have $f(y_T) - f(u) \leq \varepsilon$ after $T = O(\sqrt{\frac{L\Theta}{\varepsilon}})$ iterations.
  Almost the same as Nesterov's.

- GD: $O(\frac{LR^2}{2})$ v.s. MD: $O(\frac{\rho^2\Theta}{\varepsilon^2})$ v.s. AGD: $O(\sqrt{\frac{L\Theta}{\varepsilon}})$