Quick LP Duality

1 Linear Programming Duality

Linear programming duality underlies much of what we have been doing in class so far. In today's lecture we will formally introduce duality and relate it to the toll congestion and maximum weight matching problems from the previous lectures.

A pair of primal and dual linear programs written in the standard form is given below, we will show that any feasible solution for the dual program gives a lower bound on the value of the primal.

Primal LP	Dual LP
min $c.x$	max $y^T b$
$Ax \geq b$	$y^T A \leq c$
$x \geq 0$	$y \geq 0$

Let the objective value for the primal LP be α, it is easy to prove upper bounds $\alpha \leq \delta$ on the objective value by producing a feasible solution x with $c.x = \delta$. Linear programming duality is motivated by the question of proving lower bounds $\alpha \geq \delta$ on the objective value. It is helpful to expand the compact representation $Ax \geq b$ into m primal constraints on n variables,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

Suppose we multiply the i-th primal constraint by a positive numbers $y_i > 0$ for $i \in [m]$, positivity ensures that the signs of the inequalities are preserved. Adding the equations for all feasible primal solutions x we have the inequality,

$$\begin{aligned} (a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n)y_1 &+ \\ (a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n)y_2 &+ \\ \cdots\cdots\cdots \\ (a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_n)y_m &\geq y^T b \end{aligned}$$

Interchanging the order of summation, the above inequality can be written in terms of the coefficients of the x_i,

$$\begin{aligned} (a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m)x_1 &+ \\ (a_{21}y_1 + a_{22}y_2 + \cdots + a_{2m}y_m)x_2 &+ \\ \cdots\cdots\cdots \\ (a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nm}y_m)x_n &\geq y^T b \end{aligned}$$

If the coefficients of x_i in the above inequality are less than c_i then we have the bound $c.x \geq y^T b$ for all feasible feasible primal solutions x. We therefore have a lower bound $\min c.x \geq y^T b$ on the value of the primal linear program, note that the bound is of the form $\alpha \geq \delta$.

The problem of obtaining the sharpest lower bound on α using this method can be formulated as the dual linear program. The strong duality theorem asserts that the value of the primal and dual programs are actually equal, provided both the programs are bounded and have a feasible solution. From the above discussion it is clear that the value of any feasible solution to the primal is more than the optimal value while the value of any feasible solution to the dual is less than the optimal value.

As an example of the special case in the strong duality theorem, consider the unbounded primal program $\min -x_1, x_1 - x_2 \leq 100, x_1, x_2 \geq 0$. Verify that the dual program has one variable and is infeasible.

1.1 Complementary slackness

The complementary slackness conditions provide an easy way to verify that (x, y) are optimal solutions for a pair of primal dual linear progams,

Claim 1
If (x, y) are optimal solutions for a pair of primal dual linear programs,

$$\begin{aligned} y_i.(b - Ax_i) &= 0 \qquad \forall i \in [n] \\ x_j.(c_j - y^T A) &= 0 \qquad \forall j \in [m] \end{aligned}$$

Proof: As x, y are feasible solutions for the primal and the dual LP then we have,

$$c.x \geq y^T Ax \geq y^T b$$

Equality holds in the above equation if (x, y) are a pair of optimal solutions. Using the equality $y^T Ax = y^T b$ we have,

$$\sum_{i,j} y_i a_{ij} x_j = \sum_i y_i b_i \Rightarrow \sum_i y_i.(Ax_i - b) = 0$$

Each term in the above sum is non negative, so the sum can be zero only if the complementary slackness conditions hold. Similarly using the other equality $c.x \geq y^T Ax$ we obtain the second set of complementary slackness conditions. □

1.2 Congestion minimization

The minimum congestion problem can be written as a linear program, the variables are flows f_P on paths P in the graph, the constraints are,

$$\sum_{P:e\in P} f_P \leq \mu$$
$$\sum_{P\in P(s_i,t_i)} f_P = 1$$

We will see that the toll assignment problem is the dual of this linear program. Dual variables correspond to primal constraints, so there is a dual variable w_e for each edge and unconstrained dual variables c_i for each (s_i, t_i) pair. (Note that constraints corresponding to w_e are multiplied by -1 to bring the LP into the standard form and primal equality constraints correspond to unconstrained dual variables).

The objective function for the dual linear program is the inner product of the dual variables with the b_i from the primal constraints, the dual LP maximizes $\sum_{i\in[k]} c_i - \mu \sum_e w_e$. There is one dual constraint corresponding to each primal variable, the primal variables are flows f_P where the paths P are between (s_i, t_i) pairs. The dual constraints for paths P between (s_i, t_i) are given by,

$$\forall P \in P(s_i, t_i), \qquad c_i - \sum_{e\in P} w_e \leq 0$$

The constraints show that the maximum possible value of c_i is $w(p_i)$, the length of the shortest path between (s_i, t_i) when edges have weight w_e. The edge weights $w_e \geq 0$, we add a normalization constraint $\sum_e w(e) = 1$, the objective function changes to maximizing the lengths of the shortest paths under the metric w_e. The dual LP can be written as,

$$\max \sum_i w(p_i)$$
$$w_e \geq 0, \sum_e w_e = 1$$

The dual linear program finds a metric maximizing the average congestion over the network, recall that the average congestion for a metric is the average of the lengths of shortest paths between (s_i, t_i) pairs under the metric.

What do the complementary slackness say about an optimal pair of solutions for the toll congestion problem? The primal complementary slackness conditions $w_e(\sum_{e\in P} f_P - \mu) = 0$ tell us that only the maximally congested edges in the optimal routing can have non zero tolls assigned to them. The dual complementary slackness conditions $f_P.(c_i - \sum_{e\in P} w_e)$ tell us that only shortest paths according to the toll metric can have non zero flows in the optimal routing.

Complementary slackness conditions tell us that all flows are along shortest paths and all tolls are on the most congested paths in the optimal solution. The row player in the experts algorithm for toll congestion emulates this strategy, the experts framework is a mechanical way to obtain algorithms from LP duality.

1.3 Maximum weighted matching

Exercise: Write down the primal and dual linear programs for the cover matching problem from lecture 3, what do the complementary slackness conditions say about the maximum weight matching?