CS270 Problem Set 1 Due: Thursday March 2.

Rules: You can work in groups and consult papers and books. But you should write your paper by yourself, and you should give credit in it where credit is due, sources and colleagues. Please also write briefly, neatly, and correctly.

1. Given a drunken sailor that is walking on a pier of length n. In each step, he either increases his distance from 0 by 1 or decreases it by 1. If he is at 0, he always goes to 1.

   (a) Show that he falls off the pier in n^2 steps, with at least some constant probability. (Find a constant that works.)

   (b) Given your constant from the first part, how long must you wait to ensure he falls off the pier with probability at least $1 - 1/n$?

2. **Min-Cut**

   Show that the number of minimum cuts whose size is within a factor of α of the size of a minimum global cut is at most $n^{2\alpha}$.

3. **Flows.**

   A blocking flow in a directed flow graph is a feasible flow function (satisfies capacity and conservation constraints) where every directed $s - t$ path contains a saturated edge (i.e., $f(e) = c(e)$.) We note that in class the $c(e)$ was actually a residual capacity.

   (a) Give an example where a blocking flow is not a maximum flow.

   (b) Give an $O(m)$ time blocking flow algorithm given a directed acyclic graph where all edge capacities are one.

   (c) Consider a data structure for maintaining directed rooted trees that implements the following procedures.

   MakeTree(u) initializes a tree containing u.

   Join($(u, v), c$) requires that u is the root of a tree T and joins its tree to the tree containing v using the directed edge (u, v) associated capacity c with the edge.

   Cut(u, v) requires that (u, v) is an edge in some tree T and makes two trees by cutting the edge (u, v), one is rooted at u, the other is the previous tree without the subtree corresponding to (u, v). Returns the capacity of the cut edge.

   FindRoot(u) returns the root of the tree containing u.

   FindBottleneck(u) finds the minimum capacity edge on the path in the tree T containing u, from u to the root of T.

   ReduceCapacity(u) reduces the minimum capacity edge on the path in the tree T containing u, from u to the root of T.

   This is called a dynamic trees data structure. Each operation can be implemented in $O(\log n)$ time.

   Give psuedocode that finds a blocking flow in $O(m)$ dynamic tree operations.

   (d) Consider the dynamic tree operations above. If instead, one wishes to implement a a data structure where one only needs to maintain dynamic paths, i.e, each Join meets the condition that the an edge is only joined at the end of a a path. Briefly *sketch* an implementation of a data structure that will implement each operation in $O(\log n)$ time.

4. **Fattest Augmenting Path.**

   (a) Give an $O((n+m)\log n)$ time algorithm for finding the fattest s-t path in a weighted graph. Where the "fattest path" is defined to be the s-t path that maximizes the value of the smallest edge on the path.

   (b) Show that by implementing the Ford Fulkerson algorithm by augmenting along the fattest path in the residual network finds the optimal solution with $O(m \log |f|)$ augmenting paths. (Hint: show you route at least $1/m$ of the "residual" flow with each augmenting path. In particular, show how to decompose a flow function into m paths using depth first search on the flow graph.)

5. **Tramp Steamer**

   Consider a steamer that makes a profit of $p(i,j)$ and takes $t(i,j)$ days in going from port i to port j. Let n be the number of ports, and m be the number of trip segments, give an algorithm to determine if there is a route that makes at least P profit per day. (Hint: reduce it to determining if there is a negative cycle in a particular graph.)

6. Given a graph, G, a potential function, $p : V \to Z$ is a mapping of the vertices to integers. Given integer edge weights $c : E \to Z$, we defined the reduced cost under the potential function $p(\cdot)$ of an edge $e = (u,v)$ to be $c_p(u,v) = c(u,v) - p(v) + p(u)$.

   (a) Given that the length of the shortest path under $c_p(\cdot,\cdot)$ between u and v has value W, what is the length of the shortest path between u and v under $c(\cdot,\cdot)$? Why?

   (b) A potential function is feasible if all the reduced costs are nonnegative. Give an $O(nm)$ time algorithm to find a feasible potential function if one exists. (Hint: one does not exist if there is a negative cycle.)

   (c) Give an $O(nm + k(n\log n + m))$ algorithm for doing k single source shortest path computations in a graph with negative edge weights. (Hint: you may assume that Dijkstra's algorithm runs in time $O(n \log n + m)$.)

7. **Scheduling**

   The phone company needs to schedule telephone operators during a 24 hour cycle. They know the number of operators that they need for any single hour as specified by $r_1, \ldots, r_2 4$. Each operator works for a continuous 8 hour shift. Can the find a schedule for operators that uses at most K operators? Reduce this problem to the problem of deterimining that there is no negative cycle in a graph. (Hint: define a variable s_i to be the number of operators that start at the beginning of hour i. How many operators are on at time t in terms of the s_i? Express a constraint in terms of the s_i's and r_i's. Define a new variable $S_j = \sum_{i \leq j} s_i$. Restate the previous constraints in terms of S_j. Notice that there are only two variables in each constraint now. Show how to solve the set of equations using a shortest path computation.)

8. **Linear Programming**

   (a) Given a polytope determined by set of equations $\{Ax \leq b\}$, give an algorithm for finding the largest ball that fits inside the polytope by writing a linear program that determines the center of the ball.

(b) Given a flow problem where, in addition, one is given costs $c : E$ on the edges. Express the problem of finding a flow of value f of minimum cost as a linear program.

(c) Argue that the optimal solution to the problem above has an integer valued optimal solution.

(d) Write the dual of the "min-cost" flow problem above. Show how to use the dual solution to compute the length of the shortest s-t path when the capacities of all the arcs are f.

9. **Project**

   You may have groups of 2 to 3 people if you like and do one of the following. This is not due at this time, but form your group and choose a paper.

   (a) Choose an interesting recent or classical theory paper, understand it, and present it to the instructor (and possibly to the class.) I can make suggestions on either.

   (b) Choose any theory paper and improve the results in it. Produce a short writeup of it.

   (c) Choose any paper with an interesting algorithm or concept, and prove a theorem about it. Produce a short writeup of it.