CS–174 Combinatorics & Discrete Probability, Fall 99

Sample Midterm 1

Read these instructions carefully

1. This is a **closed book** *exam. Calculators* **are** *permitted.*

2. This midterm consists of 10 questions. The first seven questions are short answer questions and should require only a short calculation.

3. Write your answers to the other questions in the spaces provided below them. None of these questions requires a long answer, so you should have enough space; if not, continue on the back of the page and state clearly that you have done so. **Show all your working**.

4. The questions vary in difficulty: if you get stuck on some part of a question, leave it and go on to the next one.

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1. A fair 6-sided die is tossed repeatedly. Compute the expected number of tosses until two different outcomes are observed.

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2. An absent-minded secretary places n letters in n addressed envelopes in a completely random manner. What is the probability asymptotically that none of the letters is placed in the right envelope.

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3. What is the order of the largest number of balls that can be tossed into n bins, if the probability that any bin contains more than one ball is to be kept small.

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4. The random variable X has the Binomial distribution with parameters n and p.

(a) What is the expectation of X.

(b) What is the variance of X.

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5. A non-negative random variable X has expectation $\mathrm{E}[X] = 1$ and variance $\mathrm{Var}[X] = 3$. Circle those three of the following statements that must be true about X:

$\Pr[X \geq 10] \leq \frac{1}{10}$	$\Pr[X \geq 2] > 0$	$\Pr[X \leq 1] = \Pr[X \geq 1]$
$\Pr[X = 1] > 0$	$\mathrm{E}[X^2] = 4$	$\Pr[X \geq 3] = 0$

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6. n balls are tossed at random into n bins.

(a) Show that the probability that the first two bins are empty, is e^{-2}, as $n \to \infty$.

(b) Compute the probability that the first bin contains exactly one ball and the second bin is empty as $n \to \infty$.

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7. In a group of 10 people, 6 can play the piano, 5 can play the saxophone, 4 can play the violin, 4 can play the piano and the saxophone, 3 can play the piano and the violin, 2 can play the saxophone and the violin, only 1 can play all three instruments.

(a) How many people can play at least one instrument?

(b) How many can play exactly two instruments?

[continued overleaf]

8. The ABC of 123

Alice, Bob and Charlie want to choose one of the numbers 1,2 and 3 with equal probabilities. All they have is a biased coin that comes up heads with probability p, where $0 < p < 1$.

Alice suggests the following method. Toss the coin twice. If the two flips are HH, output 1, if they are HT, output 2, if they are TH, output 3. If the flips are TT, repeat the experiment.

Bob suggests the following method. Toss the coin twice. Output the number of heads obtained, plus 1.

Charlie suggets the following method. Toss the coin three times. If heads is obtained only in the i-th toss, where $1 \leq i \leq 3$, output i. Otherwise, repeat the experiment.

(a) For which values of p, if any, does Alice's method produce the numbers 1,2 and 3 with equal probabilities?

(b) For which values of p, if any, does Bob's method produce the numbers 1,2 and 3 with equal probabilities?

(c) For which values of p, if any, does Charlie's method produce the numbers 1,2 and 3 with equal probabilities?

[continued overleaf]

(d) What is the expected number of tosses used by Alice's method.

(e) What is the variance of the number of tosses used by Charlie's method.

(f) Suppose that you have a supply of biased coins, each with an unknown, and possibly different, bias. You are allowed to use each coin only twice. Can you use the coins to select one of the numbers 1,2 and 3 with equal probabilities? Show how or explain why this is not possible.

[continued overleaf]

9. A true story

A certain start-up company announces a piece of software that will compress any gigabyte of data by at least 10%. (I.e., given any string of 1Gb $= 2^{33}$ bits, it will return a compressed version of length at most 0.9×2^{33} bits. Of course, the original data must be recoverable from the compressed version. No claim is made about the performance on strings of other lengths.)

(a) Explain, at a high level, why you should be skeptical about this product. (You need not do any calculations here.)

(b) How would you construct a gigabyte of data which the software (probably) cannot compress? (Again, you need not do any calculations. Note that you **cannot** assume anything about how the company's software works.)

(c) Give an upper bound on the probability that the data you constructed in part (b) will be correctly compressed by the software. [Hint: Show that the number of different bit strings of length at most m is $\leq 2^{m+1}$.]