# Model Reduction Near Periodic Orbits of Hybrid Dynamical Systems

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**Abstract**

We show that, near periodic orbits, a class of hybrid models can be reduced to or approximated by smooth continuous–time dynamical systems. Specifically, near an exponentially stable periodic orbit undergoing isolated transitions in a hybrid dynamical system, nearby executions generically contract superexponentially to a constant–dimensional subsystem. Under a non–degeneracy condition on the rank deficiency of the associated Poincaré map, the contraction occurs in finite time regardless of the stability properties of the orbit. Hybrid transitions may be removed from the resulting subsystem via a topological quotient that admits a smooth structure to yield an equivalent smooth dynamical system. We demonstrate reduction of a high–dimensional underactuated mechanical model for terrestrial locomotion, derive a normal form for the stability basin of a hybrid oscillator, assess structural stability of event–triggered deadbeat controllers for rhythmic locomotion and manipulation, and construct stable hybrid zero dynamics. These applications illustrate the utility of our theoretical results for synthesis and analysis of feedback control laws for rhythmic hybrid behavior.

## I. Introduction

Rhythmic phenomena are pervasive, appearing in physical situations as diverse as legged locomotion on land [1], dexterous manipulation in manufacturing [2], gene regulation in cells [3], and power generation in electrical systems [4]. The most natural dynamical models for these systems are piecewise–defined or discontinuous owing to intermittent changes in the mechanical contact state of a locomotor or manipulator, or to instantaneous switches in protein synthesis or constraint activation in a gene or power network. Such *hybrid* systems generally exhibit dynamical behaviors that are distinct from those of *smooth* systems [5]. Restricting our attention to the dynamics near periodic orbits in hybrid dynamical systems, we demonstrate that a class of hybrid models for rhythmic phenomena reduce to classical dynamical systems.

Although the results of this paper do not depend on the phenomenology of the physical system under investigation, a principal application domain for this work is terrestrial locomotion. Numerous architectures have been proposed to explain how animals control their limbs; for steady–state locomotion, most posit a principle of coordination, synergy, symmetry or synchronization, and there is a surfeit of neurophysiological data to support these hypotheses [6], [7], [8], [9], [10]. Taken together, the empirical evidence suggests that the large number of degrees–of–freedom (DOF) available to a locomotor can collapse during regular motion to a low–dimensional dynamical attractor (a *template*) embedded within a higher–dimensional model (an *anchor*) that respects the locomotor's physiology [1], [11]. We provide a mathematical framework to model this empirically–observed dimensionality reduction.

From a modeling viewpoint, a stable hybrid periodic orbit provides a natural abstraction for the dynamics of steady–state legged locomotion. This approach has been widely adopted, generating a variety of models of bipedal [12], [13], [14], [15] and multi–legged [16], [17], [18] locomotion as well as some control–theoretic techniques for composition [19], coordination [20], and stabilization [21], [22], [23] of such models. In certain cases, it has been possible to embed a low–dimensional abstraction in a higher–dimensional physically–realistic model [24]. Applying these techniques to establish existence of a reduced–order subsystem imposes stringent assumptions on the dynamics of locomotion that are difficult to verify

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for any particular locomotor. In contrast, the results of this paper imply that hybrid dynamical systems generically exhibit dimension reduction near periodic orbits solely due to the interaction of the discrete–time switching dynamics with the continuous–time flow.

Under the hypothesis that iterates of the Poincaré map associated with a periodic orbit in a hybrid dynamical system are constant rank, we demonstrate the existence of a constant–dimensional invariant subsystem that attracts all nearby trajectories in finite time regardless of the stability properties of the orbit; this appears as Theorem 1 of Section III-C, below. Assuming instead that the periodic orbit under investigation is exponentially stable, we show in Theorem 2 of Section III-D that trajectories *generically* contract superexponentially to a subsystem whose dimension is determined by the rank of the linearized Poincaré map at a single point. The resulting subsystems possess a special structure that we exploit in Theorem 3 to construct a topological quotient that removes the hybrid transitions and admits the structure of a smooth manifold, yielding an equivalent smooth dynamical system.

In Section IV we show how these results can be applied to reduce the complexity of hybrid models for mechanical systems and analyze rhythmic hybrid control systems. The example in Section IV-A demonstrates that reduction can occur spontaneously in mechanical systems undergoing plastic impacts. In Section IV-B we present a family of $(3+2n)$–DOF lateral–plane multi–leg models that provably reduce to a common 3–DOF mechanical system independent of the number of limbs, $n \in \mathbb{N}$; this demonstrates model reduction in the mechanical component of the class of neuromechanical models considered in [1], [18]. As further applications, we derive a normal form for the stability basin of a hybrid oscillator in Section IV-C, assess structural stability of *event–triggered controllers* [25] for rhythmic locomotion and manipulation in Section IV-D, and provide a technique that renders *hybrid zero dynamics* [21] super–exponentially stable in Section IV-E.

## II. Preliminaries

We assume familiarity with differential topology and geometry [26], [27], and summarize notation and terminology in this section for completeness.

If $(X, \|\cdot\|)$ is a Banach space, we let $B_\delta(x) \subset X$ denote the open ball of radius $\delta > 0$ centered at $x \in X$; For $X = \mathbb{R}^n$, we may emphasize the dimension $n$ by writing $B^n_\delta(0) \subset \mathbb{R}^n$ for the open $\delta$–ball. A subset of a topological space is *precompact* if it is open and its closure is compact. A *neighborhood* of a point $x \in X$ in a topological space $X$ is an open subset $U \subset X$ containing $x$. The *disjoint union* of a collection of sets $\{S_j\}_{j\in J}$ is denoted $\coprod_{j\in J} S_j = \bigcup_{j\in J} S_j \times \{j\}$, a set we endow with the natural piecewise–defined topology. If $\sim\, \subset D \times D$ is an equivalence relation on the topological space $D$, then we let $D/\!\sim$ denote the corresponding set of equivalence classes. There is a natural *quotient projection* $\pi : D \to D/\!\sim$ sending $x \in D$ to its equivalence class $[x] \in D/\!\sim$, and we endow $D/\!\sim$ with the (unique) finest topology making $\pi$ continuous (see Appendix A in [27]). Any map $R : G \to D$ defined over a subset $G \subset D$ determines an equivalence relation $\sim\, = \{(x, y) \in D \times D : x \in R^{-1}(y),\ y \in R^{-1}(x),\ \text{or } x = y\}$. To be explicit that the equivalence relation is determined by $R$ we denote the quotient space as

$$D/\!\sim\, = \frac{D}{G \overset{R}{\sim} R(G)}.$$

A $C^r$ *$n$–dimensional manifold $M$ with boundary $\partial M$* is an $n$–dimensional topological manifold covered by an *atlas* of *$C^r$ coordinate charts* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ where $U_\alpha \subset M$ is open, $\varphi_\alpha : U_\alpha \to H^n$ is a homeomorphism, and $H^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n \geq 0\}$ is the upper half–space; we write $\dim M = n$. The charts are $C^r$ in the sense that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a $C^r$ diffeomorphism over $\varphi_\beta(U_\alpha \cap U_\beta)$ for all pairs $\alpha, \beta \in \mathcal{A}$ for which $U_\alpha \cap U_\beta \neq \emptyset$; if $r = \infty$ we say $M$ is *smooth*. The boundary $\partial M \subset M$ contains those points that are mapped to the plane $\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n = 0\}$ in some chart. A map $P : M \to N$ is $C^r$ if $M$ and $N$ are $C^r$ manifolds and for every $x \in M$ there is a pair of charts $(U, \varphi), (V, \psi)$ with $x \in U \subset M$ and $P(x) \in V \subset N$ such that the coordinate representation $\widetilde{P} = \psi \circ P \circ \varphi^{-1}$ is a smooth map between subsets of $H^n$. We let $C^r(M, N)$ denote the Banach space of $C^r$ maps between $M$ and $N$ (see Chapter 2 in [26]).

Each $x \in M$ has an associated *tangent space* $T_xM$, and the disjoint union of the tangent spaces is the *tangent bundle* $TM = \coprod_{x\in M} T_xM$. Note that any element in $TM$ may be regarded as a pair $(x, v)$ where $x \in M$ and $v \in T_xM$, and $TM$ is naturally a smooth $2n$–dimensional manifold. We let $\mathfrak{T}(M)$ denote the set of *smooth vector fields* on $M$, i.e. smooth maps $F : M \to TM$ for which $F(x) = (x, v)$ for some $v \in T_xM$ and all $x \in M$. It is a fundamental result that any $F \in \mathfrak{T}(M)$ determines an ordinary differential equation in every chart on the manifold that may be solved globally to obtain a *maximal flow* $\phi : \mathcal{F} \to M$ where $\mathcal{F} \subset \mathbb{R} \times M$ is the *maximal flow domain* (see Theorem 17.8 in [27]).

If $P : M \to N$ is a smooth map between smooth manifolds, then at each $x \in M$ there is an associated linear map $DP(x) : T_xM \to T_{P(x)}N$ called the *pushforward*. Globally, the pushforward is a smooth map $DP : TM \to TN$; in coordinates, it is the familiar Jacobian matrix. If $M = X \times Y$ is a product manifold, the pushforward naturally decomposes as $DP = (D_xP,\, D_yP)$ corresponding to derivatives taken with respect to $X$ and $Y$, respectively. The *rank* of a smooth map $P : M \to N$ at a point $x \in M$ is $\operatorname{rank} DP(x)$. If $\operatorname{rank} DP(x) = r$ for all $x \in M$, we simply write $\operatorname{rank} DP \equiv r$; if $P$ is furthermore a homeomorphism onto its image, then $P$ is a *smooth embedding*, and the image $P(M)$ is a *smooth embedded submanifold*. The difference $\dim N - \dim P(M)$ is called the *codimension* of $P(M)$. In this case, any smooth vector field $F \in \mathfrak{T}(M)$ may be pushed forward to a unique smooth vector field $DP(F) \in \mathfrak{T}(P(M))$. A vector field $F \in \mathfrak{T}(M)$ is *inward–pointing* at $x \in \partial M$ if for any coordinate chart $(U, \varphi)$ with $x \in U$ the $n$–th coordinate of $D\varphi(F)$ is positive and *outward–pointing* if it is negative.

## III. Hybrid Dynamical Systems

We describe a class of hybrid systems useful for modeling physical phenomena in Section III-A, then restrict our attention to the behavior of such systems near periodic orbits in Section III-B. It was shown in [28] that the Poincaré map associated with a periodic orbit of a hybrid system is generally not full rank; we explore the geometric consequences of this rank loss. Under a non–degeneracy condition on this rank loss we demonstrate in Section III-C that the hybrid system possesses an invariant hybrid subsystem to which all nearby trajectories contract in finite time regardless of the stability properties of the orbit. In Section III-D we show that the invariance and contraction of the subsystem hold approximately for any exponentially stable hybrid periodic orbit. Using tools from differential topology, we remove hybrid transitions from the resulting reduced–order subsystems in Section III-E to yield a continuous–time dynamical system that governs the behavior of the hybrid system near its periodic orbit.

### A. Hybrid Differential Geometry

For our purposes, it is expedient to define hybrid dynamical systems over a finite disjoint union $M = \coprod_{j\in J} M_j$ where $M_j$ is a connected manifold with boundary for each $j \in J$; we endow $M$ with the natural (piecewise–defined) topology and smooth structure. We refer to such spaces as *smooth hybrid manifolds*. Note that the dimensions of the constituent manifolds are not required to be equal. Several differential–geometric constructions naturally generalize to such spaces; we prepend the modifier 'hybrid' to make it clear when this generalization is invoked. For instance, the *hybrid tangent bundle* $TM$ is the disjoint union of the tangent bundles $TM_j$, and the *hybrid boundary* $\partial M$ is the disjoint union of the boundaries $\partial M_j$.

Let $M = \coprod_{j\in J} M_j$ and $N = \coprod_{\ell\in L} N_\ell$ be two hybrid manifolds. Note that if a map $R : M \to N$ is continuous, then for each $j \in J$ there exists $\ell \in L$ such that $R(M_j) \subset N_\ell$ and hence $R|_{M_j} : M_j \to N_\ell$. Using this observation, there is a natural definition of differentiability for continuous maps between hybrid manifolds. Namely, a map $R : M \to N$ is called *smooth* if $R$ is continuous and $R|_{M_j} : M_j \to N$ is smooth for each $j \in J$. In this case the *pushforward* $DR : TM \to TN$ is the smooth map defined piecewise as $DR|_{TM_j} = D(R|_{M_j})$ for each $j \in J$. The rank of a smooth map $R : M \to N$ is defined using its pushforward as in Section II. A smooth map $F : M \to TM$ is called a *vector field* if for all $x \in M$ there exists $v \in T_xM$ such that $F(x) = (x, v)$.

With these preliminaries established, we can define the class of hybrid systems considered in this paper. This is a specialization of *hybrid automata* [5] that emphasizes the differential–geometric character of hybrid phenomena.

**Definition 1.** *A* hybrid dynamical system *is specified by a tuple $H = (D, F, G, R)$ where:*
$D = \coprod_{j \in J} D_j$ *is a smooth hybrid manifold;*
$F : D \to TD$ *is a smooth vector field;*
$G \subset \partial D$ *is an open subset of $\partial D$;*
$R : G \to D$ *is a smooth map.*
*As in [5], we call $R$ the* reset map *and $G$ the* guard*. When we wish to be explicit about the order of smoothness, we will say $H$ is $C^r$ if $D$, $F$, and $R$ are $C^r$ as a manifold, vector field, and map, respectively, for some $r \in \mathbb{N}$.*

Roughly speaking, an *execution* of a hybrid dynamical system is determined from an initial condition in $D$ by following the continuous–time dynamics determined by the vector field $F$ until the trajectory reaches the guard $G$, at which point the reset map $R$ is applied to obtain a new initial condition.

**Definition 2.** *An* execution *of a hybrid dynamical system $H = (D, F, G, R)$ is a right–continuous function $x : T \to D$ over an interval $T \subset \mathbb{R}$ such that:*
1) *if $x$ is continuous at $t \in T$, then $x$ is differentiable at $t$ and $\frac{d}{dt}x(t) = F(x(t))$;*
2) *if $x$ is discontinuous at $t \in T$, then the limit $x(t^-) = \lim_{s \to t^-} x(s)$ exists, $x(t^-) \in G$, and $R(x(t^-)) = x(t)$.*

If $F$ is tangent to $G$ at $x \in G$, there is a possible ambiguity in determining a trajectory from $x$ since one may either follow the flow of $F$ on $D$ or apply the reset map to obtain a new initial condition $y = R(x)$.

**Assumption 1.** *$F$ is outward–pointing on $G$.*

**Remark 1.** *The use of time–invariant vector fields and reset maps in Definition 1 is without loss of generality in the following sense. Suppose $D$ is a hybrid manifold, $G \subset \partial D$ is open, and $F : \mathbb{R} \times D \to TD$, $R : \mathbb{R} \times G \to D$ define a time–varying vector field and reset map, respectively. Define*

$$\widetilde{D} = \mathbb{R} \times D, \ \widetilde{G} = \mathbb{R} \times G,$$

*and let $\widetilde{F} : \widetilde{D} \to T\widetilde{D}$, $\widetilde{R} : \widetilde{G} \to \widetilde{D}$ be defined in the obvious way. Then $\widetilde{H} = (\widetilde{D}, \widetilde{F}, \widetilde{G}, \widetilde{R})$ is a hybrid dynamical system in the form of Definition 1.*

### *B. Hybrid Periodic Orbits and Hybrid Poincaré Maps*

In this paper, we are principally concerned with *periodic* executions of hybrid dynamical systems, which are nonequilibrium trajectories that intersect themselves.

**Definition 3.** *An execution $\gamma : T \to D$ is* periodic *if there exists $s \in T$, $\tau > 0$ such that $s + \tau \in T$ and*

$$\gamma(s) = \gamma(s + \tau). \tag{1}$$

*If there is no smaller positive number $\tau$ such that* (1) *holds, then $\tau$ is called the* period *of $\gamma$, and we will say $\gamma$ is a $\tau$–*periodic orbit.

**Remark 2.** *The domain $T$ of a periodic orbit may be taken to be the entire real line, $T = \mathbb{R}$, without loss of generality. In the sequel we conflate the execution $\gamma : \mathbb{R} \to D$ with its image $\gamma(\mathbb{R}) \subset D$.*

Motivated by the applications in Section IV, we restrict our attention to periodic orbits undergoing *isolated transitions*, i.e. a finite number of discrete transitions that occur at distinct time instants. In addition to excluding *Zeno* periodic orbits [29] from our analysis, this assumption enables us to construct

Poincaré maps (see [30], [31] for the classical case) associated with the hybrid periodic orbit $\gamma$. Roughly speaking, a *Poincaré map* $P : U \to \Sigma$ is defined over an open subset $U \subset \Sigma$ of an embedded codimension–1 submanifold $\Sigma \subset D$ that intersects the periodic orbit at exactly one point $\{\xi\} = \gamma \cap \Sigma$ by tracing an execution from $x \in U$ forward in time until it intersects $\Sigma$ at $P(x)$. The submanifold $\Sigma$ is referred to as a *Poincaré section*. It is known that this procedure yields a map that is well–defined and smooth near the fixed point $\xi = P(\xi)$ [13], [28], [32]. Unlike the classical case, Poincaré maps in hybrid systems need not be full rank.

It is a standard result for continuous–time dynamical systems that the eigenvalues of the linearization of the Poincaré map at its fixed point—commonly called *Floquet multipliers*—do not depend on the choice of Poincaré section (see Section 1.5 in [31]). This fact generalizes to the hybrid setting in the sense that non–zero eigenvalues are shared between Poincaré maps defined on different sections [28]. A straightforward application of Sylvester's inequality [33] shows that the rank of the Poincaré map is bounded above by the minimum dimension of all hybrid domains. More precise bounds are pursued elsewhere [28], but the following Proposition will suffice for the Applications in Section IV.

**Proposition 1.** *If $P : U \to \Sigma$ is a Poincaré map associated with a periodic orbit $\gamma$, then $\forall x \in U$ : $\operatorname{rank} DP(x) \leq \min_{j\in J} \dim D_j - 1$.*

### C. Exact Reduction

When iterates of the Poincaré map associated with a periodic orbit of a hybrid dynamical system have constant rank, executions initialized nearby converge in finite time to a constant–dimensional subsystem.

**Theorem 1** (Exact Reduction). *Let $\gamma$ be a periodic orbit that undergoes isolated transitions in a hybrid dynamical system $H = (D, F, G, R)$, $P : U \to \Sigma$ a Poincaré map for $\gamma$, $n = \min_j \dim D_j - 1$, and suppose there exists a neighborhood $V \subset U$ of $\{\xi\} = \gamma \cap U$ and $r \in \mathbb{N}$ such that $\operatorname{rank} DP^n(x) = r$ for all $x \in V$. Then there exists an $(r+1)$–dimensional hybrid embedded submanifold $M \subset D$ and a hybrid open set $W \subset D$ for which $\gamma \subset M \cap W$ and trajectories starting in $W$ contract to $M$ in finite time.*

*Proof:* Applying Lemma 3 from Appendix A-B1 to $P$, there is a neighborhood $V \subset U$ of $\{\xi\} = \gamma \cap U$ such that $S = P^n(V)$ is an $r$–dimensional embedded submanifold of $U \subset \Sigma$, $P|_S$ maps $S$ diffeomorphically onto $P(S)$, and $P(S) \cap S$ is an open subset of $S$. Without loss of generality we assume $U \subset G \cap \partial D_1$ and the periodic orbit $\gamma$ passes through each domain once per cycle. Set $\gamma_1 = \gamma \cap G \cap \partial D_1$, let $U_2 \subset D_2$ be a neighborhood of $R(\gamma_1)$ over which Lemma 1 from Appendix A-A1 may be applied to construct a time–to–impact map $\sigma_2 : U_2 \to \mathbb{R}$, let $G_1 = R^{-1}(U_2)$ be a neighborhood of $\gamma_1$ in $G \cap \partial D_1$, and let $\phi_1 : \mathcal{F}_1 \to D_1$ the maximal flow of $F|_{D_1}$ on $D_1$. Proceed inductively forward around the cycle to construct, for each $j \in J$: the exit point $\gamma_j = \gamma \cap G \cap \partial D_j$; time–to–impact map $\sigma_j : U_j \to \mathbb{R}$ over a neighborhood $U_j \subset D_j$ containing $R(\gamma_{j-1})$; a neighborhood $G_j = R^{-1}(U_{j+1}) \subset G \cap \partial D_j$ containing $\gamma_j$; and the maximal flow $\phi_j : \mathcal{F}_j \to D_j$ of $F|_{D_j}$ on $D_j$.

By flowing $S$ forward through one cycle, for each $j \in J$ we will construct a submanifold $M_j \subset D_j$ that is diffeomorphic to $[0, 1] \times \mathbb{R}^r$. Observe that, since $P|_S$ is a diffeomorphism, with $S_1 = S \cap G_1$ we have that the restriction $R|_{S_1}$ is a diffeomorphism onto its image and $F|_{R(S_1)}$ is nowhere tangent to $R(S_1)$. Let $M_2 \subset D_2$ be the embedded submanifold obtained by flowing $R(S_1)$ to $G \cap \partial D_2$, and let $S_2 = M_2 \cap G_2$; observe that $S_2$ is diffeomorphic to $S_1$, $M_2$ is diffeomorphic to $[0, 1] \times S_2$, and $F|_{D_2}$ is tangent to $M_2$. Proceed inductively forward around the cycle to construct, for each $j \in J$, an embedded submanifold $S_j \subset G_j$ diffeomorphic to $S_1$ and a submanifold $M_j \subset D_j$ diffeomorphic to $[0, 1] \times S_j$ such that $F|_{D_j}$ is tangent to $M_j$. Note that $S_1$ is diffeomorphic to the $r$–dimensional manifold $\mathbb{R}^r$, so $\dim M_j = r + 1$ for each $j \in J$. The subsystem $M = \coprod_{j\in J} M_j \subset D$ contains $\gamma$, is invariant under the continuous flow by construction, and is invariant under the reset map in the sense that $R^{-1}(M) \cap M \subset G \cap M$ is open.

Finally, let $W_1 = \phi_j^{-1}(\mathbb{R} \times V) \subset D_1$ be the open set that flows into $V$, where $S = P^n(V)$ was defined in the first paragraph of the proof. Let $W_{|J|} = \phi_{|J|}^{-1}(R^{-1}(W_1)) \subset D_{|J|}$ be the open set that flows into $W_1$ where $|J|$ denotes the number of elements in $J$ (i.e. the number of hybrid domains). Proceed inductively backward around the cycle to construct, for each $j \in J$, an open set $W_j \subset D_j$ that flows into $S$ in finite time. Then the hybrid open set $W = \coprod_{j \in J} W_j \subset D$ contains $\gamma$ and all executions initialized in $W$ flow into $S \subset M$ in finite time. ■

Since $M$ is invariant under the continuous dynamics (i.e. $F|_M$ is tangent to $M$) and the discrete dynamics (i.e. $R(G \cap M) \subset M$), it determines a hybrid subsystem that determines the stability of $\gamma$ in $H$.

**Corollary 1.** *$H|_M = (M, F|_M, G \cap M, R|_{G \cap M})$ is a hybrid dynamical system with periodic orbit $\gamma$.*

**Corollary 2.** *The periodic orbit $\gamma$ is Lyapunov (resp. asymptotically, exponentially) stable in $H$ if and only if $\gamma$ is Lyapunov (resp. asymptotically, exponentially) stable in $H|_M$.*

When the rank at the fixed point $\xi = P(\xi)$ achieves the upper bound stipulated by Proposition 1, the following Corollary ensures that $DP^n$ is constant rank (and hence Theorem 1 may be applied). This is important since it is possible to compute a lower bound for $\operatorname{rank} DP^n(\xi)$ via numerical simulation [34].

**Corollary 3.** *If* $\operatorname{rank} DP^n(\xi) = \min_{j \in J} \dim D_j - 1$*, then there exists an open set $V \subset U$ containing $\xi$ such that* $\operatorname{rank} DP^n(x) = \min_{j \in J} \dim D_j - 1$ *for all $x \in V$.*

The choice of Poincaré section in Theorem 1 is irrelevant in the sense that the Poincaré map $\widetilde{P} : \widetilde{U} \to \widetilde{\Sigma}$ defined over any other Poincaré section $\widetilde{\Sigma}$ will be constant–rank in a neighborhood $\widetilde{V} \subset \widetilde{U}$ of its fixed point $\left\{\widetilde{\xi}\right\} = \gamma \cap \widetilde{\Sigma}$, as the following Corollary shows; this follows directly from Lemma 4 in [32].

**Corollary 4.** *If $\widetilde{P} : \widetilde{U} \to \widetilde{\Sigma}$ is a Poincaré map for $\gamma$ with fixed point $\widetilde{\xi} = \widetilde{P}(\widetilde{\xi})$, then there exists an open subset $\widetilde{V} \subset \widetilde{U}$ containing $\widetilde{\xi}$ such that* $\operatorname{rank} D\widetilde{P}^n(x) = r$ *for all $x \in \widetilde{V}$.*

### *D. Approximate Reduction*

By restricting our attention to exponentially stable periodic orbits, we find that a hybrid system generically contracts superexponentially to a constant–dimensional subsystem near a periodic orbit.

**Theorem 2** (Approximate Reduction)**.** *Let $\gamma$ be an exponentially stable periodic orbit undergoing isolated transitions in a hybrid dynamical system $H = (D, F, G, R)$, $P : U \to \Sigma$ a Poincaré map for $\gamma$, $n = \min_j \dim D_j - 1$, and $r = \operatorname{rank} DP^n(\gamma \cap \Sigma)$. Then there exists an $(r+1)$–dimensional hybrid embedded submanifold $M \subset D$ such that for any $\varepsilon > 0$ there exists a hybrid open set $W^\varepsilon \subset D$ for which $\gamma \subset M \cap W^\varepsilon$ and the distance from trajectories starting in $W^\varepsilon$ to $M$ contracts by $\varepsilon$ each cycle.*

*Proof:* Without loss of generality we assume $U \subset G \cap \partial D_1$ and the periodic orbit $\gamma$ passes through each domain once per cycle. For each $j \in J$ let $P_j : U_j \to \Sigma_j$ be a Poincaré map for $\gamma$ defined over $U_j \subset \Sigma_j \subset G \cap \partial D_j$, and let $\{\gamma_j\} = \gamma \cap G \cap \partial D_j$ be the exit point of $\gamma$ in $D_j$. By a straightforward application of Sylvester's inequality (see Appendix A.5.4 in [33]), we find $\operatorname{rank} DP_j^n(\gamma_j) = r$ for all $j \in J$. Applying Lemma 4 from Appendix A-B2 implies that for each $j \in J$ there exists an open set $V_j \subset U$ containing $\gamma_j$ and a $C^1$ diffeomorphism $\varphi_j : V_j \to \mathbb{R}^{n_j - 1}$ where $n_j = \dim D_j$ such that $\varphi_j(\gamma_j) = 0$ and the coordinate representation $\widetilde{P}_j = \varphi_j \circ P_j \circ \varphi_j^{-1}$ of $P_j$ has the form $\widetilde{P}_j(z_j, \zeta_j) = (A_j z_j,\ N_j(z_j, \zeta_j))$ where $z_j \in \mathbb{R}^r$, $\zeta_j \in \mathbb{R}^{n_j - 1 - r}$, $A_j \in \mathbb{R}^{r \times r}$ is invertible, $N_j(0,0) = 0$, and $D_{\zeta_j} N_j(0,0)$ is nilpotent.

Fix $j \in J$ and let $S_j = \varphi_j^{-1}(\mathbb{R}^r \times \{0\}) \subset V_j$ be the $r$–dimensional embedded submanifold tangent to the non–nilpotent eigendirections of $DP_j^n(\gamma_j)$. Observe that $DR|_{G \cap S_j}(\gamma_j)$ has rank $r = \dim S_j$, hence by the Inverse Function Theorem (see Theorem 7.10 in [27]) there is a neighborhood $N_j \subset S_j$ containing $\gamma_j$ such that $R|_{N_j} : N_j \to D$ is a diffeomorphism onto its image $R(N_j) \subset D_{j+1}$. Furthermore, since $\operatorname{rank} DP_j^n(\gamma_j) = r$, the vector field is transverse to $R(N_j)$ at $\gamma_j$, i.e. $F(R(\gamma_j)) \notin T_{R(\gamma_j)} R(N_j)$, and we assume $N_j$ was chosen small enough so that $F$ is transverse along all of $R(N_j)$. Let $M_{j+1} \subset D_{j+1}$ be the embedded submanifold obtained by flowing $R(N_j)$ forward to $G$; note that $M_{j+1}$ is diffeomorphic to $[0,1] \times \mathbb{R}^r$. Observe that $M = \coprod_{j \in J} M_j$ is invariant under the continuous flow (i.e. $F|_M$ is tangent to $M$) and approximately invariant under the reset map in the sense that $DR|_{G \cap M}$ is tangent to $M$ on $\gamma$:

for all $j \in J$ and $v \in T_{\gamma_j}(G \cap M)$ we have $DR|_{G\cap M}(\gamma_j)v \in T_{R(\gamma_j)}M$. For each $j \in J$, let $\Pi_j : V_j \to G$ be a smooth map defined as follows. Given $x \in V_j$, write $(z_x, \zeta_x) = \varphi_j(x) \in \mathbb{R}^r \times \mathbb{R}^{n_j - r - 1}$ and let $\Pi_j(x) = \varphi_j^{-1}(z_x, 0)$. Observe that $R \circ \Pi_j|_{G\cap M_j} : G \cap M_j \to M_{j+1}$ is a diffeomorphism onto its image.

Fix $\varepsilon > 0$ and apply the construction in the proof of Lemma 5 from Appendix A-B3 to obtain a radius $\delta > 0$ and for each $j \in J$ a norm $\|\cdot\|_j^\varepsilon : \mathbb{R}^{n_j-1} \to \mathbb{R}$ such that the nonlinearity $\widetilde{P}_j(z_j, \zeta_j) - (A_j z_j, 0)$ contracts exponentially fast with rate $\varepsilon$ on $B_\delta^{n_j-1}(0) \subset \mathbb{R}^{n_j-1}$ as measured by $\|\cdot\|_j^\varepsilon$. For each $j \in J$ define $V_j^\varepsilon = \varphi_j^{-1}(B_\delta^{n_j-1}(0)) \subset G \cap \partial D_j$, let $\phi_j : \mathfrak{F}_j \to D_j$ denote the maximal flow of $F|_{D_j}$ on $D_j$, and let $W_j^\varepsilon = \phi_j^{-1}(\mathbb{R} \times V_j^\varepsilon) \subset D_j$ be the (open) set of points that flow into $V_j^\varepsilon$. Since $\phi_j$ is the flow of a smooth vector field transverse to $V_j^\varepsilon$, any $x \in W_j^\varepsilon$ can be written uniquely as $x = \phi_j(t_x, v_x)$ for some $t_x \leq 0$ and $v_x \in V_j^\varepsilon$. Using this representation, we endow $W_j^\varepsilon$ with a distance metric $d_j^\varepsilon : W_j^\varepsilon \times W_j^\varepsilon \to \mathbb{R}$ by defining $d_j^\varepsilon(x, y) = |t_x - t_y| + \|\varphi_j(v_x) - \varphi_j(v_y)\|_j^\varepsilon$. Observe that the exponential contraction of $\widetilde{P}_j$ at rate $\varepsilon$ in $\|\cdot\|_j^\varepsilon$ to $\varphi_j(M_j \cap G)$ implies exponential contraction of executions initialized in $W_j^\varepsilon$ at rate $\varepsilon$ to $M$ in $d_j^\varepsilon$.

Finally, let $W^\varepsilon = \coprod_{j\in J} W_j^\varepsilon$ and $M^\varepsilon = M \cap W^\varepsilon$. Define a smooth hybrid map $\Pi^\varepsilon : G \cap W^\varepsilon \to G$ piecewise for each $j \in J$ by observing that $G \cap W_j^\varepsilon \subset V_j$ and letting $\Pi^\varepsilon(x) = \Pi_j(x)$ for all $x \in G \cap W_j^\varepsilon$. ■

**Corollary 5.** *Letting $M^\varepsilon = M \cap W^\varepsilon$, the collection $H|_{M^\varepsilon} = (M^\varepsilon, F|_{M^\varepsilon}, G \cap M^\varepsilon, R \circ \Pi^\varepsilon|_{G\cap M^\varepsilon})$ is a $C^1$ hybrid dynamical system with periodic orbit $\gamma$, where $\Pi^\varepsilon : G \cap W^\varepsilon \to G$ is the smooth hybrid map constructed in the proof of Theorem 2.*

Although the submanifold $M \subset D$ is invariant under the continuous dynamics of $H$ in the sense that $F|_M$ is tangent to $M$, the reset map must be modified to ensure $M$ is invariant under the discrete dynamics. However, since $DR|_{G\cap M^\varepsilon} = D\left(R \cap \Pi^\varepsilon\right)|_{G\cap M^\varepsilon}$, the map $\Pi$ does not affect $R$ to first order.

**Remark 3.** *We emphasize that the rank hypothesis on the Poincaré map $P : U \to \Sigma$ in Theorem 2, that $\operatorname{rank} DP^n(\gamma \cap \Sigma) = r$ at the point $\{\xi\} = \gamma \cap \Sigma$, is weaker than the hypothesis in Theorem 1, that $\operatorname{rank} DP^n(x) = r$ for all $x$ in an open set $V \subset U$. In particular, approximating the rank over an uncountably infinite set typically involves estimates on higher–order derivatives of $P^n$.*

### *E. Smoothing*

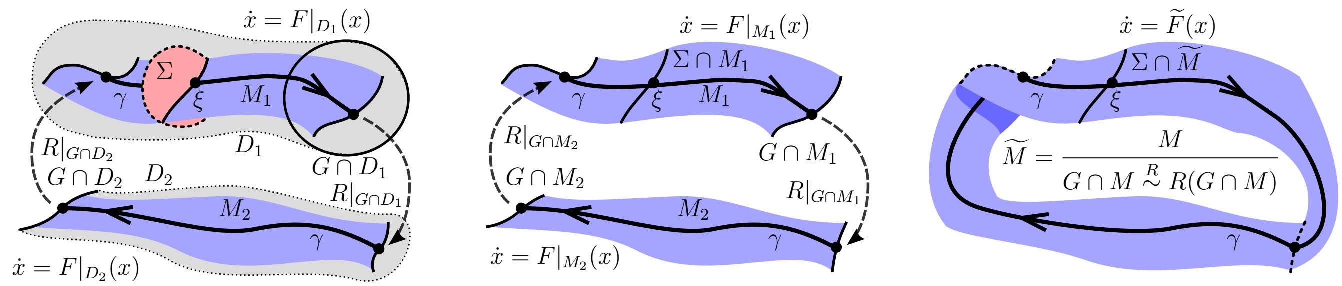
The subsystems yielded by Theorems 1 and 2 on *exact* and *approximate* reduction share important properties: the constituent manifolds have the same dimension; the reset map is a hybrid diffeomorphism between disjoint portions of the boundary; and the vector field points inward along the range of the reset map. Under these conditions, we can globally *smooth* the hybrid transitions using techniques from differential topology to obtain a single continuous–time dynamical system. Executions of the hybrid (sub)system are preserved as integral curves of the continuous–time system. This provides a smooth $n$–dimensional generalization of the *hybrifold* construction in [35].

**Theorem 3** (Smoothing)**.** *Let $H = (M, F, G, R)$ be a hybrid dynamical system with $M = \coprod_{j\in J} M_j$. Suppose $\dim M_j = n$ for all $j \in J$, $R(G) \subset \partial M$, $\partial M = G \coprod R(G)$, $R$ is a hybrid diffeomorphism onto its image, and $F$ is inward–pointing along $R(G)$. Then the topological quotient $\widetilde{M} = \frac{M}{G \overset{R}{\sim} R(G)}$ may be endowed with the structure of a smooth manifold such that:*

1) *the quotient projection $\pi : M \to \widetilde{M}$ restricts to a smooth embedding $\pi|_{M_j} : M_j \to \widetilde{M}$ for each $j \in J$;*
2) *there is a smooth vector field $\widetilde{F} \in \mathfrak{T}(\widetilde{M})$ such that any execution $x : T \to M$ of $H$ descends to an integral curve of $\widetilde{F}$ on $\widetilde{M}$ via $\pi : M \to \widetilde{M}$:*

$$\forall t \in T : \frac{d}{dt}\pi \circ x(t) = \widetilde{F}\left(\pi \circ x(t)\right).$$

*Proof:* Let $S \subset G \cap M_i$ be a connected component in some domain $i \in J$, and let $k \in J$ be the index for which $R(S) \subset M_k$. The hypotheses of this Theorem together with Assumption 1 ensure Lemma 2




(a) $H = (D, F, G, R)$ (b) $H|_M = (M, F|_M, G \cap M, R|_{G\cap M})$ (c) $(\widetilde{M}, \widetilde{F})$

Fig. 1. (a) Applying Theorem 1 (Exact Reduction) to a hybrid dynamical system $H = (D, F, G, R)$ containing a periodic orbit $\gamma$ with associated Poincaré map $P : U \to \Sigma$ yields an invariant subsystem $M = \coprod_{j\in J} M_j$; nearby trajectories contract to $M$ in finite time. (b) The subsystem may be extracted to yield a hybrid dynamical system $H|_M$. (c) The hybrid system $H|_M$ may subsequently be smoothed via Theorem 3 (Smoothing) to yield a continuous-time dynamical system $(\widetilde{M}, \widetilde{F})$. Application of Theorem 3 to the subsystem yielded by Theorem 2 (Approximate Reduction) is illustrated by replacing $H|_M$ in (b) by $H|_{M^\varepsilon}$.

from Appendix A-A2 may be applied to attach $M_i$ to $M_k$ to yield a new smooth manifold $\widetilde{M}_{ik}$. The hybrid system defined over domain $\widetilde{M}_{ik} \coprod_{j\neq i,k} M_j$ and guard $G \setminus S$ satisfies the hypotheses of this Theorem, hence we may inductively attach domains on each connected component that remains in $G \setminus S$. This yields a smooth manifold $\widetilde{M}$ and vector field $\widetilde{F} \in \mathfrak{T}(\widetilde{M})$ with the required properties. ■

**Remark 4.** *As illustrated in Fig. 1, Theorem 3 is applicable to the subsystems $H|_M$, $H|_{M^\varepsilon}$ that emerge as a consequence of the Corollaries to Theorems 1 and 2, respectively. Thus a class of hybrid models for periodic phenomena may be reduced (exactly or approximately) to smooth dynamical systems.*

## IV. Applications

The Theorems of Section III apply directly to autonomous hybrid dynamical systems; in Section IV-A we demonstrate that reduction to a smooth subsystem can occur spontaneously in a mechanical system undergoing intermittent impacts. The results are also applicable to systems with control inputs; in Section IV-B we synthesize a state–feedback control law that reduces a family of multi–leg models for lateral–plane locomotion to a common low–dimensional subsystem. Further, the reduction of hybrid dynamics to a smooth subsystem provides a route through which tools from classical dynamical systems theory can be generalized to the hybrid setting; in Section IV-C we extend a normal form for limit cycles. Finally, we analyze the structural stability of event–triggered deadbeat control laws for locomotion in Section IV-D and suggest a technique for stabilization of hybrid–invariant subsystems in Section IV-E.

### A. *Spontaneous Reduction in a Vertical Hopper*

In this section, we apply Theorem 1 (Exact Reduction) to the *vertical hopper* example shown in Fig. 2. This system evolves through an *aerial* mode and a *ground* mode. The aerial mode $D_a$ consists of the set of configurations where the lower mass is above the ground (see Fig. 2 and its caption for notation), $(y, \dot{y}, x, \dot{x}) \in D_a = T\mathbb{R} \times T\mathbb{R}_{\geq 0}$. The vector field $F|_{D_a}$ is given by Newton's laws, $\mu\ddot{y} = k(\ell - (y - x)) - \mu g$, $m\ddot{x} = -k(\ell - (y - x)) - b\dot{x} - mg$. The boundary $\partial D_a = \{(y, \dot{y}, x, \dot{x}) \in D_a : x = 0\}$ contains the states where the lower mass has just impacted the ground, and a hybrid transition occurs on the subset $G_a = \{(y, \dot{y}, 0, \dot{x}) \in \partial D_a : \dot{x} < 0\}$ of the boundary $D_a$ where the lower mass has negative velocity. In this case, the state is reinitialized in the ground mode by annihilating the velocity of the lower mass, i.e. $R|_{G_a} : G_a \to D_g$ is defined by $R|_{G_a}(y, \dot{y}, 0, \dot{x}) = (y, \dot{y})$. In the ground mode, the lower mass is pressed into the ground but has no dynamics, and the boundary consists of the set of configurations where the force in the aerial mode allows the mass to lift off: $D_g = \{(y, \dot{y}) \in T\mathbb{R} : -k(\ell - y) \leq mg\}$, $\partial D_g = \{(y, \dot{y}) \in D_g : -k(\ell - y) = mg\}$. The vector field $F|_{D_g}$ is given by $\mu\ddot{y} = ak(\ell - y) - \mu g$. A hybrid

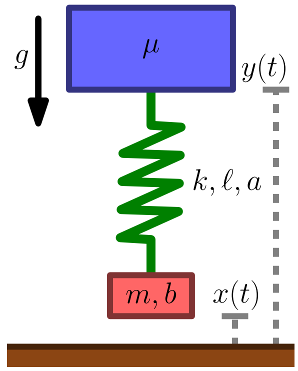



Fig. 2. Schematic of vertical hopper. Two masses $m$ and $\mu$, constrained to move vertically above a ground plane in a gravitational field with magnitude $g$, are connected by a linear spring with stiffness $k$ and nominal length $\ell$. The lower mass experiences viscous drag proportional to velocity with constant $b$ when it is in the air, and impacts plastically with the ground (i.e. it is not permitted to penetrate the ground and its velocity is instantaneously set to zero whenever a collision occurs). When the lower mass is in contact with the ground, the spring stiffness is multiplied by a constant $a > 1$.

transition occurs when the forces balance and will instantaneously increase to pull the mass off the ground, $G_g = \{(y, \dot{y}) \in \partial D_g : \dot{y}(t) > 0\}$, and the state is reset in the aerial mode by initializing the position and velocity of the lower mass to zero, i.e. $R|_{G_g} : G_g \to D_a$ is defined by $R|_{G_g}(y, \dot{y}) = (y, \dot{y}, 0, 0)$. This defines a hybrid dynamical system $(D, F, G, R)$ where

$$D = D_a \coprod D_g, \; F \in \mathfrak{T}(D), \; G = G_a \coprod G_g, \; R : G \to D.$$

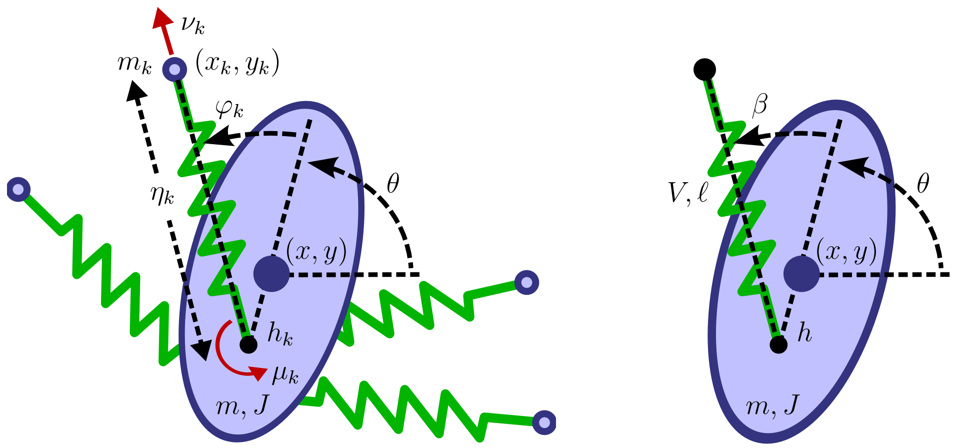
With parameters $(m, \mu, k, b, \ell, a, g) = (1, 3, 10, 5, 2, 2, 2)$, the vertical hopper possesses a stable periodic orbit $\gamma = (y^*, \dot{y}^*, x^*, \dot{x}^*)$ to which nearby trajectories $(y, \dot{y}, x, \dot{x})$ converge asymptotically. Choosing a Poincaré section $\Sigma$ in the ground domain $D_g$ at mid-stance, $\Sigma = \{(y, \dot{y}) : \dot{y} = 0\} \subset D_g$, we find numerically[1] using parameter values given in the caption of Fig. 2 that the hopper possesses a stable periodic orbit $\gamma$ that intersects the Poincaré section at $\gamma \cap \Sigma = \{\xi\}$ where $\xi = (y, \dot{y}) \approx (0.94, 0.00)$. The linearization $DP$ of the associated scalar–valued Poincaré map $P : \Sigma \to \Sigma$ has eigenvalue $\operatorname{spec} DP(\xi) \approx 0.57$ at the fixed point $P(\xi) = \xi$. The rank of the Poincaré map $P$ attains the upper bound of Proposition 1, hence Corollary 3 implies the rank hypothesis of Theorem 1 (Exact Reduction) is satisfied. Thus the dynamics of the hopper collapse to a one degree–of–freedom mechanical system after a single hop. Algebraically, the physical (holonomic) constraint that activates when the lower mass impacts the ground transfers to the aerial mode where no such physical constraint exists.

### *B. A smooth feedback law that reduces a polyped with $(3 + 2n)$ DOF to a 3 DOF LLS*

In this section, we synthesize a state–feedback control law under which the underactuated lateral–plane polyped illustrated in Fig. 3a exactly reduces to the Lateral Leg–Spring (LLS) [17] model in Fig. 3b. With $n$ limbs, the polyped possesses $(3 + 2n)$ degrees–of–freedom (DOF); the LLS has 3 DOF. This example, though valuable for connecting disparate models of locomotion [17], [18], is intended primarily to demonstrate how the Theorems of Section III can be used to synthesize controllers that reduce the dimension of a mechanical system by an arbitrary degree.

The LLS is an energy–conserving lateral–plane model for locomotion comprised of: an inertial body with kinetic energy $\frac{1}{2}\dot{q}^T M \dot{q}$ where $q = (x, y, \theta) \in Q = \mathbb{R}^2 \times S^1 = \mathrm{SE}(2)$ gives the planar position and orientation of the body and $M = \operatorname{diag}(m, m, J) \in \mathbb{R}^{3\times 3}$ the mass distribution of the body; and a massless leg–spring that attaches at hip position $h \in \mathbb{R}^2$ with potential energy $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and rest

[1]For numerical simulations, we use a recently–developed algorithm [34] with step size $h = 1\times 10^{-2}$ and relaxation parameter $\epsilon = 1\times 10^{-10}$. The sourcecode will be made available online.




(a) polyped with $n = 4$ legs; leg $\#k$ is annotated

(b) Lateral Leg–Spring (LLS) [17]

Fig. 3. Illustration of lateral–plane models for locomotion described in Section IV-B.

length $\ell > 0$ (i.e. $V(\ell) = 0$). The system is initialized at the start of a stride by orienting the leg at an angle $\beta \in S^1$ with respect to the body and touching the foot down on the left side of the body at position $f = (x, y)^T + r(\theta)\left(h + r(\beta)(\ell, 0)^T\right) \in \mathbb{R}^2$, where $r(\theta) \in \mathrm{SO}(2)$ is the matrix that performs counterclockwise rotation by an angle $\theta \in S^1$. It is assumed that the leg will instantaneously contract: with $\eta = \|(x, y) + r(\theta)h - f\|$ denoting the leg length, we have $\dot{\eta} < 0$. The leg pivots frictionlessly about $f$ until the leg extends to its rest length: $\eta = \ell$ and $\dot{\eta} > 0$. At this point the foot is instantaneously reset to touch down on the right side of the body at position $f = (x, y)^T + r(\theta)\left(h + r(-\beta)(\ell, 0)^T\right)$, where once again it is assumed $\dot{\eta} < 0$ for the new foot location. The stride ends once the leg again extends to its rest length; subsequent strides are defined inductively. In world coordinates, any execution of the LLS defined for $t$ units of time yields a body trajectory $(q, \dot{q}) : [0, t] \to TQ$ in the configuration manifold $TQ$.

Defining a Poincaré section $\Sigma$ at the touchdown event that initiates a stride, we see that $\Sigma$ is parameterized in the body frame of reference by the magnitude of the forward speed $v \in \mathbb{R}_{\geq 0}$ and the angle of the velocity vector with respect to the body $\delta \in S^1$ (see Fig. 3 in [17]). Since the total energy $E \in \mathbb{R}$ is conserved, the angular velocity at touchdown $\omega \in \mathbb{R}$ can be determined from $v$ and $E$. In an appropriate parameter regime [17], the Poincaré map $P : \Sigma \to \Sigma$ possesses an exponentially stable fixed point $(v^*, \delta^*) = P(v^*, \delta^*)$ that corresponds to a straight running gait.

We now describe the underactuated hybrid control system illustrated in Fig. 3a. This extends the neuromechanical models proposed to study multi–legged locomotion in [1], [18] by introducing masses into $n \in \mathbb{N}$ feet connected by massless limbs attached at fixed hip locations $\{h_k\}_{k=1}^n$ on the inertial body. We assume that each foot can be attached or detached from the substrate at any time, and the transition from *swing* to *stance* annihilates the kinetic energy in a foot. There are two actuators that act on limb $k$: a hip torque $\mu_k$ and a prismatic force $\nu_k$. For simplicity we assume the inputs do not saturate so that any $\mu_k \in \mathbb{R}, \nu_k \in \mathbb{R}$ are feasible at any configuration of the mechanical system. The $n$–leg polyped's dynamics have the form

$$\begin{aligned} M\ddot{q} &= \sum_{k=1}^{n} \mu_k D_q \varphi_k(q, q_k), + \nu_k D_q \eta_k(q, q_k) \\ m_k \ddot{q}_k &= \mu_k D_{q_k} \varphi_k(q, q_k) + \nu_k D_{q_k} \eta_k(q, q_k) \end{aligned} \tag{2}$$

for each $k \in \{1, \ldots, n\}$ where $q_k = (x_k, y_k)$ denotes the position of the $k$–th foot,

$$\begin{aligned} \varphi_k(q, q_k) &= \arctan\left((x, y) + r(\theta)h_k - (x_k, y_k)\right) - \theta, \\ \eta_k(q, q_k) &= \|(x, y) + r(\theta)h_k - (x_k, y_k)\| \end{aligned}$$

give the angle and length of the $k$–th leg, and $D_q\varphi_k$ denotes the Jacobian derivative of the function $\varphi_k$ with respect to the coordinates contained in $q$. Partitioning the foot index set into $\mathrm{stance}\coprod\mathrm{swing}$, let

$$f_{\mathrm{stance}} = \sum_{k\in\mathrm{stance}} \mu_k D_q\varphi_k(q,q_k) + \nu_k D_q\eta_k(q,q_k),$$
$$f_{\mathrm{swing}} = \sum_{k\in\mathrm{swing}} \mu_k D_q\varphi_k(q,q_k) + \nu_k D_q\eta_k(q,q_k)$$

denote the net wrench [36] on the body from stance limbs and swing limbs, respectively. Let $U, V \in \mathbb{R}^{3\times|\mathrm{stance}|}$ be the matrices whose $k$–th columns are given by

$$U_k = D_q^T\varphi(q,q_k),\ \ V_k = D_q^T\eta(q,q_k), \tag{3}$$

for each $k \in \mathrm{stance}$. Then so long as the matrix $(U,V)$ has full row rank any wrench $f \in T^*Q$ may be imposed on the body by appropriate choice of inputs to the stance limbs $\mu,\nu \in \mathbb{R}^{|\mathrm{stance}|}$.

To reduce the $n$–leg polyped, $n \geq 4$, to the LLS from an initial condition $(q,\dot{q})(0) \in TQ$ over the time interval $[0,t] \subset \mathbb{R}$, we proceed as follows.

1) Let $(q,\dot{q}) : [0,t] \to TQ$ be the LLS execution from initial condition $(q,\dot{q})(0)$ over the time interval $[0,t]$. Set $t_0 = 0$, and let $t_1 \in [0,t]$ denote the time of the first discrete transition.
2) Let $i = 1$ and partition the legs into $\mathrm{swing}$ and $\mathrm{stance}$, ensuring $|\mathrm{swing}| \geq 2, |\mathrm{stance}| \geq 2$.
3) For each $k \in \mathrm{swing}$ choose inputs $\mu_k,\nu_k$ over the time interval $[t_{i-1},t_i]$ such that $\varphi_k(t_i) = \beta_k$, $\eta_k(t_i) = \ell_k$, i.e. the swing feet are at prespecified kinematic configurations with respect to the body at the instant where the LLS undergoes a touchdown transition.[2]
4) For each $k \in \mathrm{stance}$ choose inputs $\mu_k,\nu_k$ over the time interval $[t_{i-1},t_i]$ to (a) cancel the net wrench exerted on the body by the swing limbs' inputs as computed in Step 3) and (b) impose the wrench experienced by the LLS as determined in Step 1).[3]
5) If a discrete transition occurs in the interval $(t_i,t]$, let $t_{i+1}$ denote the time of the first transition, set $i = i+1$, and return to step 3).

This procedure clearly results in a body trajectory for the polyped that exactly matches the LLS over the time interval $[0,t]$. The control inputs specified in steps 1)–4) above are smooth functions of the state $x = ((q,\dot{q}),\{(q_k,\dot{q}_k)\}_{k=1}^n) \in (T\,\mathrm{SE}(3)) \times (\prod_{k=1}^n T\mathbb{R}^2) = D$, and together with (2) define a smooth (closed–loop) hybrid vector field $F \in \mathfrak{T}(D)$. The periodic orbit for the LLS lifts to the polyped, and hence the polyped trajectories converge asymptotically to a straight running gait. After a single stride the feet obey a set of smooth holonomic constraints,

$$\forall k \in \{1,\ldots,n\} : q_k(t) = \psi_k(q(t))$$

where $\psi_k : Q \to \mathbb{R}$ is obtained by integrating the closed–loop dynamics for foot $k$ from the prespecified touchdown configuration. Thus the rank of the polyped's Poincaré map is constant and equal to the rank of the LLS's Poincaré map. Therefore Theorem 1 (Exact Reduction) implies the $n$–leg polyped exactly reduces to the LLS after one stride.

### C. *Hybrid Floquet Coordinates*

When a hybrid system reduces to a smooth dynamical system near a periodic orbit via Theorem 1 (Exact Reduction), we can generalize the *Floquet normal form* [37], [38], [39], [40] for the orbit from classical dynamical systems theory to the hybrid setting. Consider a hybrid dynamical system $H = (D,F,G,R)$ with $\tau$–periodic orbit $\gamma$ that satisfies the hypotheses of Theorem 1. Let $M \subset D$ be the $(r+1)$–dimensional invariant hybrid subsystem yielded by the Theorem, and $W \subset D$ a hybrid open set containing $\gamma$ that

[2]This can be achieved for instance by feedback linearizing $\varphi$ and $\eta$ into double integrators $\ddot{\varphi}_k = u_k$, $\ddot{\eta}_k = v_k$, then applying constant inputs $u_k = 2\left(\beta_k - \varphi_k(t_{i-1})\right)/(t_i - t_{i-1})^2$, $v_k = 2\left(\ell_k - \eta_k(t_{i-1})\right)/(t_i - t_{i-1})^2$.

[3]This can be achieved for instance by taking the least–squares projection of $f_{\mathrm{LLS}} - f_{\mathrm{swing}}$ onto $(\mu,\nu) \in \mathbb{R}^{2|\mathrm{stance}|}$ using $(U,V)$ from (3).

contracts to $M$ in finite time. Let $(\widetilde{M}, \widetilde{F})$ denote the smooth dynamical system obtained by applying Theorem 3 (Smoothing).

Under a genericity condition[4] there exists a neighborhood $U \subset \widetilde{M}$ of $\gamma$ and a smooth chart $\varphi : U \to \mathbb{R}^r \times S^1$ such that the coordinate representation of the vector field has the form

$$D\varphi \circ \widetilde{F} \circ D\varphi^{-1}(z,\theta) = \begin{pmatrix} \dot{z} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} A(\theta)z \\ 2\pi/\tau \end{pmatrix} \tag{4}$$

where $z \in \mathbb{R}^r$ and $\theta \in S^1$. In these coordinates, each $\theta \in S^1$ determines an embedded submanifold $\widetilde{N}_\theta = \mathbb{R}^r \times \{\theta\} \subset \mathbb{R}^r \times S^1$ that is mapped to itself after flowing forward in time by $\tau$; for this reason, the submanifolds $\widetilde{N}_\theta$ are referred to as *isochrons* [38]. Each $x \in \widetilde{N}_\theta$ may be assigned the *phase* $\theta \in S^1$; if $\gamma$ is stable, then as $t \to \infty$ the trajectory initialized at $x$ will asymptotically converge to the trajectory initialized at $(0, \theta)$.

The isochrons may be pulled back to any precompact hybrid open set $V \subset W$ containing $\gamma$ in the original hybrid system as follows. The proof of Theorem 1 implies there exists a finite time $t < \infty$ such that every execution initialized in $V$ is defined over the time interval $[0, t]$ and reaches $M$ before time $t$; without loss of generality, we take this time to be a multiple $k\tau$ of the period of $\gamma$ for some $k \in \mathbb{N}$. Let $\psi : V \to \widetilde{M}$ denote the map that flows an initial condition $x \in V$ forward by $t$ time units and then applies the quotient projection $\pi : M \to \widetilde{M}$ obtained from Theorem 3 to yield the point $\psi(x) \in \widetilde{M}$. Then the constructions in the proof of Theorem 1 imply that $\psi$ is a smooth map in the sense defined in Section III-A, i.e. it is continuous and $\psi|_{V \cap D_j}$ is smooth for each $j \in J$. Now for any $\theta \in S^1$ the set $N_\theta = \psi^{-1}(U)$ is mapped into $\widetilde{N}_\theta$ after $k\tau$ units of time; we thus refer to $N_\theta \subset D$ as a *hybrid isochron*. We conclude by noting that $N_\theta$ will generally not be a smooth (hybrid) submanifold.

### D. Structural Stability of Event–Triggered Deadbeat Control

Consider a hybrid system wherein a finitely–parameterized control input updates when the system passes intermittently through a distinguished subset of state space. This form of event–triggered control [25] in rhythmic hybrid systems dates back (at least) to Raibert's hoppers [41] and Koditschek's jugglers [2], and has received recent interest [22], [42]. We model this with a hybrid system $H = (D, F, G, R)$ whose vector field and reset map accept a control input that takes values in a smooth boundaryless $m$–dimensional manifold $\Theta$. The value of the control input may be updated whenever an execution passes through the guard $G$, but it does not change in response to the continuous flow. We study the *structural stability* (see Section 1.7 in [31]) of attracting invariant sets arising in this class of systems by applying the Theorems of Section III.

Suppose for some $\theta \in \Theta$ that $H$ possesses a periodic orbit $\gamma$. Let $P : U \times \Theta \to \Sigma$ be a Poincaré map associated with $\gamma$ where $U \subset \Sigma \subset G$, and let $\{\xi\} = \gamma \cap \Sigma$. As noted in [22], a straightforward application of the Implicit Function Theorem (see Theorem 7.8 in [27]) shows that if $\operatorname{rank} D_\theta P(\xi, \theta) = \dim \Sigma$ then there exists a neighborhood $V \subset U$ of $\xi$ and a smooth feedback law $\psi : V \to \Theta$ such that for all $x \in V$ we have $P(x, \psi(x)) = \xi$, i.e. $\psi$ is a *deadbeat* feedback control law. Since $\psi$ is smooth, this closed–loop Poincaré map satisfies the hypotheses of Theorem 1 (Exact Reduction) with rank $r = 0$, so the invariant hybrid subsystem yielded by the Theorem is simply the periodic orbit $\gamma$. Attracting invariant submanifolds of intermediate dimension can be constructed using this technique; applying only the first step of the "two–step" controller in [22] is one example. Up to this point in this section, we have simply recapitulated the results of [22] in the context of our Theorem 1 (Exact Reduction); we proceed by applying Theorem 2 (Approximate Reduction) to study the *structural stability* of invariant sets arising from such deadbeat control laws.

Suppose the preceding development is applied to a model that differs from that used to construct the feedback law $\psi \in C^\infty(V, \Theta)$. If the models differ by a small smooth deformation (as would occur

[4]Either the periodic orbit is exponentially stable or it is *hyperbolic* and the associated *Floquet multipliers* do not satisfy any *Diophantine equation* (see Chapter 3.3 in [31]).

if there was a small perturbation in model parameters), one interpretation of this change is that some $\widetilde{\psi} \in B_\varepsilon(\psi) \subset C^\infty(V, \Theta)$ is applied to the model for which $\psi$ is deadbeat, where $\varepsilon > 0$ bounds the error. For all $\varepsilon > 0$ sufficiently small, $\widetilde{\psi}$ yields a perturbed closed–loop Poincaré map $\widetilde{P} : V \to \Sigma$ possessing a unique fixed point $\widetilde{\xi} \in V$, and $\left\| D\widetilde{P}(\widetilde{\xi}) \right\|_i < L\varepsilon$ where $L$ is a local Lipschitz bound for $P$ on $V$. Therefore $\widetilde{\xi}$ is an exponentially stable fixed point of the perturbed system, and nearby trajectories contract at a rate proportional to $\varepsilon$. Though this analysis is known for smooth maps, we believe the application to hybrid dynamical systems is novel.

We conclude by noting that it is possible for the structure of the hybrid dynamics to constrain the achievable perturbations. For instance, if one domain of the hybrid system has lower dimension than that in which the Poincaré map is constructed, then zero is always a Floquet multiplier regardless of the applied feedback; in this case Theorem 2 (Approximate Reduction) implies the existence of a proper submanifold of the Poincaré section $\Sigma$ to which trajectories contract superexponentially.

### *E. Novel Technique for Constructing and Stabilizing Hybrid Zero Dynamics*

In the paradigm of *hybrid zero dynamics* [21], feedback control is employed to asymptotically zero an output map $\eta : D \to \mathbb{R}^k$ defined over the state space $D$ of a hybrid dynamical system $H = (D, F, G, R)$. The output function is chosen carefully so that its zero section, $Z = \eta^{-1}(0) \subset D$, is a hybrid embedded submanifold of the state space that is invariant under the continuous dynamics (i.e. $F|_Z$ is tangent to $Z$) and discrete dynamics (i.e. $R(G \cap Z) \subset Z$). Then a periodic orbit $\gamma \subset Z$ that is stable in the *hybrid zero dynamics* $H|_Z = (Z, F|_Z, G \cap Z, R|_{G\cap Z})$ can be stabilized in the original hybrid system $H$ by ensuring nearby trajectories contract toward $Z$ sufficiently quickly.

The results of this paper provide techniques for stabilization, analysis, and construction of hybrid zero dynamics. In [13], finite–time convergence to a two–dimensional hybrid zero dynamics subsystem was achieved with a continuous state feedback law that is not locally Lipschitz continuous (and hence not $C^1$) [43]. In contrast, our Theorem 1 (Exact Reduction) shows that finite–time convergence to hybrid zero dynamics is achieved for the class of *smooth* state feedback control laws constructed in [22] as described in Section IV-D. Subsequently [21], [44] developed smooth feedback laws that rendered a hybrid zero dynamics subsystem exponentially stable. Since the resulting Poincaré map is a diffeomorphism, these closed–loop systems may be *smoothed* via Theorem 3 (Smoothing) to yield equivalent smooth dynamical systems. Finally, Theorem 2 (Approximate Reduction) suggests a mechanism to construct hybrid zero dynamics that are *approximately* invariant. Specifically, as in Section IV-D let $P : U \times \Theta \to \Sigma$ be a Poincaré map for a hybrid periodic orbit corresponding to a fixed point $\xi = P(\xi, 0)$ where $\Theta$ is smooth manifold that parameterizes a control input for the hybrid system and $U \subset \Sigma \subset G$. By linearizing $P$ about its fixed point $\xi = P(\xi, 0)$, we obtain a linear discrete–time control system

$$x_{i+1} = D_x P(\xi, 0) x_i + D_\theta P(\xi, 0) u_i. \tag{5}$$

Any linear subspace of the controllable subspace (see Chapter 8d in [33]) of (5) may be rendered attracting in finite time via a linear state feedback law. By applying this linear controller to the original nonlinear hybrid system, Theorem 2 implies the existence of an approximately–invariant subsystem, tangent to the target subspace at the fixed point $\xi = P(\xi, 0)$, that attracts nearby trajectories superexponentially.

## V. DISCUSSION

Generically for an exponentially stable periodic orbit in a hybrid dynamical system, nearby trajectories contract superexponentially to a subsystem containing the orbit. Under a non–degeneracy condition on the rank of any Poincaré map associated with the orbit, this contraction occurs in finite time regardless of the stability of the orbit. Hybrid transitions may be removed from the resulting subsystem, yielding an equivalent smooth dynamical system. Thus the dynamics near stable hybrid periodic orbits are generally obtained by extending the behavior of a smooth system in transverse coordinates that decay superexponentially. Although the applications presented in Section IV focused on terrestrial locomotion [1], we emphasize that the results in Section III do not depend on the phenomenology of the physical system under investigation, and are hence equally suited to study rhythmic hybrid control systems appearing in robotic manipulation [2], biochemistry [3], and electrical systems [4].

In addition to providing a canonical form for the dynamics near hybrid periodic orbits, the results of this paper suggest a mechanism by which a many–legged locomotor or a multi–fingered manipulator may collapse a large number of mechanical degrees–of–freedom to produce a low–dimensional coordinated motion. This provides a link between disparate lines of research: formal analysis of hybrid periodic orbits; design of robots for rhythmic locomotion and manipulation tasks; and scientific probing of neuromechanical control architectures in humans and animals. It shows that hybrid models of rhythmic phenomena generically reduce dimensionality, that this reduction may be deliberately designed into an engineered system, and hence that evolution may have exploited this reduction in developing its spectacularly dexterous agents.

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## Appendix A
## Smooth Dynamical Systems

We constructed the hybrid systems considered in this paper using switching maps defined on boundaries of smooth dynamical systems. The behavior of such systems can be studied by alternately applying flows and maps, thus in this section we collect results that provide canonical forms for the behavior of flows and maps near periodic orbits and fixed points, respectively. The first two results concern continuous–time dynamical systems and may be found in textbooks, hence we state them without proof. The third and fourth establish a canonical form for submanifolds that are invariant and approximately invariant (respectively) near fixed points in discrete–time dynamical systems. The fifth result provides an estimate of the error in the invariance approximation.

### *A. Continuous–Time Dynamical Systems*

**Definition 4.** *A* continuous–time dynamical system *is a pair* $(M, F)$ *where:*

$M$ *is a smooth manifold with boundary* $\partial M$*;*

$F$ *is a smooth vector field on* $M$*, i.e.* $F \in \mathcal{T}(M)$.

*1) Time–to–Impact:* When a trajectory passes transversely through an embedded submanifold, the time required for nearby trajectories to pass through the manifold depends smoothly on the initial condition (see Chapter 11.2 in [30]). This provides the prototype used in the proofs of Theorems 1 and 2 for the dynamics near the portion of a hybrid periodic orbit in one domain of a hybrid system.

**Lemma 1.** *Let* $(M, F)$ *be a smooth dynamical system,* $\phi : \mathcal{F} \to M$ *the maximal flow associated with* $F$*, and* $G \subset M$ *a smooth codimension–1 embedded submanifold. If there exists* $x \in M$ *and* $t \in \mathcal{F}^x$ *such that* $\phi(t, x) \in G$ *and* $F(\phi(t, x)) \notin T_x G$*, then there is a neighborhood* $U \subset M$ *containing* $x$ *and a smooth map* $\sigma : U \to \mathbb{R}$ *so that* $\sigma(x) = t$ *and* $\phi(\sigma(y), y) \in G$ *for all* $y \in U$*;* $\sigma$ *is called the* time–to–impact *map.*

**Remark 5.** *This lemma is applicable when* $G \subset \partial M$.

*2) Smoothing Flows:* Two continuous–time dynamical systems can be smoothly attached to one another along their boundaries to obtain a new continuous–time system (see Theorem 8.2.1 in [26]). Distinct hybrid domains were attached to one another using this construction in Section III.

**Lemma 2.** *Suppose $(M_1, F_1), (M_2, F_2)$ are $n$–dimensional continuous–time dynamical systems, there exists a diffeomorphism $R : \partial M_1 \to \partial M_2$, $F_1$ is outward–pointing along $\partial M_1$, and $F_2$ is inward–pointing along $\partial M_2$. Then the topological quotient*

$$\widetilde{M} = \frac{M_1 \coprod M_2}{\partial M_1 \overset{R}{\sim} \partial M_2}$$

*can be endowed with the structure of a smooth manifold such that for $j \in \{1, 2\}$:*

1) *the quotient projections $\pi_j : M_j \to \widetilde{M}$ are smooth embeddings; and*
2) *there is a smooth vector field $\widetilde{F} \in \mathfrak{T}(\widetilde{M})$ that restricts to $D\pi_j(F_j)$ on $\pi(M_j) \subset \widetilde{M}$.*

**Remark 6.** *The smooth structure constructed in Lemma 2 is unique up to diffeomorphism (see Theorem 2.1 of Chapter 8 in [26]).*

### *B. Discrete–time Dynamical Systems*

**Definition 5.** *A* discrete–time dynamical system *is a pair $(\Sigma, P)$ where:*

$\Sigma$ *is a smooth manifold without boundary;*
$P$ *is a smooth endomorphism of $\Sigma$, i.e. $P : \Sigma \to \Sigma$.*

In studying hybrid dynamical systems, we encounter smooth maps $P : \Sigma \to \Sigma$ that are noninvertible. Viewing iteration of $P$ as determining a discrete–time dynamical system, we wish to study the behavior of these iterates near a fixed point $\xi = P(\xi)$. Note that if $P$ has constant rank equal to $k < n = \dim \Sigma$, then its image $P(\Sigma) \subset \Sigma$ is an embedded $k$–dimensional submanifold near $\xi$ by the Rank Theorem (see Theorem 7.13 in [27]). With an eye toward model reduction, one might hope that the composition $(P \circ P) : \Sigma \to P(\Sigma)$ is also constant–rank, but this is not generally true[5].

In this section we provide three results that introduce regularity into iterates of a noninvertible map $P : \Sigma \to \Sigma$ on an $n$–dimensional manifold $\Sigma$ near a fixed point $P(\xi) = \xi$. If $P^n$, the $n$–th iterate of $P$, has constant rank equal to $r \in \mathbb{N}$ near the fixed point $\xi$, then $P$ reduces to a diffeomorphism over an $r$–dimensional invariant submanifold after $n$ iterations; this result is given in Section A-B1. Even if $DP^n$ is not constant rank, as long as $\xi$ is exponentially stable then $P$ can be approximated by a diffeomorphism on a submanifold whose dimension equals rank $DP^n(\xi)$; this is the subject of Section A-B2. A bound on the error in this approximation is provided in Section A-B3.

*1) Exact Reduction:* When the derivative of the $n$–th iterate of $P : \Sigma \to \Sigma$ has constant rank near a fixed point, then the range of $P$ is locally an embedded submanifold, and $P$ restricts to a diffeomorphism over that submanifold. This originally appeared without proof as Lemma 3 in [32].

**Lemma 3.** *Let $(\Sigma, P)$ be an $n$–dimensional discrete–time dynamical system with $P(\xi) = \xi$ for some $\xi \in \Sigma$. Suppose the rank of $P$ is bounded above by $n \in \mathbb{N}$ and the composition of $P$ with itself $n$ times, $P^n$, has constant rank equal to $r \in \mathbb{N}$ on a neighborhood of $\xi$. Then there is a neighborhood $V \subset \Sigma$ containing $\xi$ such that $P^n(V)$ is an $r$–dimensional embedded submanifold near $\xi$ and there is a neighborhood $U \subset P^n(V)$ containing $\xi$ that $P$ maps diffeomorphically onto $P(U) \subset P^n(V)$.*

In the proof of Lemma 3, we make use of a fact from linear algebra obtained by passing to the Jordan canonical form.

**Proposition 2.** *If $A \in \mathbb{R}^{m\times m}$ and* $\operatorname{rank} A \le n$, *then* $\operatorname{rank}(A^{2n}) = \operatorname{rank}(A^n)$.

[5]Consider the map $P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $P(x, y) = (x^2, x)$.

*Proof: (of Lemma 3)* By the Rank Theorem (see Theorem 7.13 in [27]), there is a neighborhood $V \subset \Sigma$ of $\xi$ for which $S = P^n(V)$ is an $r$–dimensional embedded submanifold and by Proposition 2 we have

$$\begin{aligned}\operatorname{rank}\left(DP^n|_S\right)(\xi) &= \operatorname{rank} D(P^n \circ P^n)(\xi)\\ &= \operatorname{rank} DP^n(\xi).\end{aligned}$$

Therefore $DP^n|_S : T_\xi S \to T_\xi S$ is a bijection, so by the Inverse Function Theorem (see Theorem 7.10 in [27]), there is a neighborhood $W \subset S$ containing $\xi$ so that $P^n(W) \subset S$ and $P^n|_W : W \to P^n(W)$ is a diffeomorphism.

We now show that $W$ is invariant under $P$ in a neighborhood of $\xi$. By continuity of $P$, there is a neighborhood $L \subset V$ containing $\xi$ for which $P(L) \subset V$ and $P^n(L) \subset W$. The set $U = P^n(L)$ is a neighborhood of $\xi$ in $S$. Further, we have

$$P(U) = P \circ P^n(L) = P^n \circ P(L) \subset S.$$

The restriction $P^n|_U : U \to P^n(U)$ is a diffeomorphism since $U \subset W$, whence $P|_U$ is a diffeomorphism onto its image $P(U) \subset S$. ■

*2) Approximate Reduction:* Now consider the case where $P^n$ is not constant rank but $\xi = P(\xi)$ is exponentially stable, meaning that the *spectral radius* $\rho\left(DP(\xi)\right) = \max\left\{|\lambda| : \lambda \in \operatorname{spec} DP(\xi)\right\}$ is less than unity, $\rho\left(DP(\xi)\right) < 1$ (see [33] for more detail). We show that $P$ may be approximated by a diffeomorphism defined on a submanifold whose dimension equals $\operatorname{rank} DP^n(\xi)$. The technical result we desire was originally established by Hartman [45]; refer to Appendix B for more detail. We apply Hartman's Theorem to construct a $C^1$ change–of–coordinates that exactly linearizes all eigendirections corresponding to non–zero eigenvalues of $DP(\xi)$ in the following Lemma.

**Lemma 4.** *Let $(\Sigma, P)$ be an $n$–dimensional discrete–time dynamical system. Suppose $\xi = P(\xi)$ is an exponentially stable fixed point and let $r = \operatorname{rank} DP^n(\xi)$. Then there is a neighborhood $U \subset \Sigma$ of $\xi$ and a $C^1$ diffeomorphism $\varphi : U \to \mathbb{R}^n$ such that $\varphi(\xi) = 0$ and the coordinate representation $\widetilde{P} = \varphi \circ P \circ \varphi^{-1}$ of $P$ has the form*

$$\widetilde{P}(z, \zeta) = (Az,\ N(z, \zeta))$$

*where $z \in \mathbb{R}^r$, $\zeta \in \mathbb{R}^{n-r}$, $A \in \mathbb{R}^{r \times r}$ is invertible, $N : \varphi(U) \to \mathbb{R}^{n-r}$ is $C^1$, $N(0,0) = 0$, and $D_\zeta N(0,0)$ is nilpotent.*

*Proof:* Let $(U_0, \varphi_0)$ be a smooth chart for $\Sigma$ with $\xi \in U_0$ and $\varphi_0(\xi) = 0$. We begin by verifying that the hypotheses of Theorem 4 from Appendix B are satisfied for the map $P_0 : \varphi_0(U_0) \to \mathbb{R}^n$ defined by $P_0 = \varphi_0 \circ P \circ \varphi_0^{-1}$. Let $\lambda \in \operatorname{spec} DP_0(0)$ be the eigenvalue with largest magnitude, and $\ell \in \mathbb{N}$ its algebraic multiplicity. Applying the linear change–of–coordinates that puts $DP_0(0)$ into Jordan canonical form, we assume

$$DP_0(0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $B \in \mathbb{R}^{\ell \times \ell}$ and $\operatorname{spec} B = \{\lambda\}$. Now in the notation of Theorem 4,

$$P_0(x, y) = (Ax + X(x, y), By + Y(x, y))$$

where $x \in \mathbb{R}^{n-\ell}$, $y \in \mathbb{R}^\ell$, and $X$, $Y$ are smooth and $X(0,0) = 0$, $Y(0,0) = 0$; note that $m = 0$ (there is no $z$ coordinate) at this step. Because $X$ and $Y$ are smooth on the neighborhood $U_0$ of the origin, their derivatives are uniformly Lipschitz and Hölder continuous on a precompact open subset of $U_0$.

Theorem 4 implies there exists a neighborhood $U_1 \subset \mathbb{R}^n$ of the origin and a $C^1$ diffeomorphism $\varphi_1 : U_1 \to \mathbb{R}^n$ for which the map $P_1 : \varphi_1(U_1) \to \mathbb{R}^n$ defined by $P_1 = \varphi_1 \circ P_0 \circ \varphi_1^{-1}$ has the form (after reversing the order of the coordinates)

$$P_1(z_1, \zeta_1) = (A_1 z_1,\ N_1(z_1, \zeta_1))$$

where $z_1 \in \mathbb{R}^{r_1}$, $r_1 > 0$, $\zeta_1 \in \mathbb{R}^{n-r_1}$ and $A_1 \in \mathbb{R}^{r_1 \times r_1}$ is invertible. Observe that the map $P_1$ satisfies the hypotheses of Theorem 4. Therefore we may inductively apply the Theorem to construct a sequence of coordinate charts $\{(U_k, \varphi_k)\}_{k=1}^K$ and corresponding maps $\{P_k\}_{k=1}^K$ such that for all $k \in \{1, \dots, K\}$

$$P_k(z_k, \zeta_k) = (A_k z_k,\ N_k(z_k, \zeta_k))$$

where $z_k \in \mathbb{R}^{r_k}$, $\zeta_k \in \mathbb{R}^{n-r_k}$, $A_k \in \mathbb{R}^{r_k \times r_k}$ is invertible, and $r_k > r_{k-1}$ (note that $r_0 = 0$). The sequence terminates at a finite $K < \infty$ with $r_K = r = \operatorname{rank} DP^n(\xi)$. Therefore in the $C^1$ chart $(U, \varphi)$ given by $\varphi = \varphi_K \circ \dots \circ \varphi_0$ and $U = \varphi^{-1}(\mathbb{R}^n)$, the coordinate representation $\widetilde{P} = \varphi \circ P \circ \varphi^{-1}$ of $P$ has the form

$$\widetilde{P}(z, \zeta) = (Az,\ N(z, \zeta))$$

where $z \in \mathbb{R}^r$, $\zeta \in \mathbb{R}^{n-r}$ and $A \in \mathbb{R}^{r \times r}$ is invertible. Since $A$ is invertible and $\operatorname{rank} D\widetilde{P}^n(\xi) = r$, $D_\zeta N(0,0)$ is nilpotent. ■

*3) Superstability:* Finally, we recall that if all eigenvalues of the linearization of a map at a fixed point are zero—a so–called "superstable" fixed point [28]—then the map contracts superexponentially.[6]

**Lemma 5.** *Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map with $P(0) = 0$, $\operatorname{spec} DP(0) = \{0\}$. Then for every $\varepsilon > 0$ and norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ there exists $\delta, C > 0$ such that*

$$\forall x \in B_\delta(0), k \in \mathbb{N} : \left\|P^k(x)\right\| \leq C\varepsilon^k \|x\|.$$

In the proof of Lemma 5, we use the following elementary fact regarding induced norms.

**Proposition 3** (1.3.6 in [46]). *Given $\varepsilon > 0$ and $A \in \mathbb{R}^{n \times n}$, there exists a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ such that $\|A\|_i \leq \rho(A) + \varepsilon$, where $\|\cdot\|_i : \mathbb{R}^{n \times n} \to \mathbb{R}$ is the operator norm induced by $\|\cdot\|$ and $\rho(A)$ is the spectral radius of $A$.*

*Proof:* (of Lemma 5) Given $\varepsilon > 0$, choose the norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ obtained by applying Proposition 3 to $DP(0)$ so that $\|DP(0)\|_i \leq \frac{1}{2}\varepsilon$. Since $DP$ is continuous, there exists a $\delta > 0$ such that

$$\forall x \in B_\delta(0) : \|DP(x) - DP(0)\|_i < \frac{1}{2}\varepsilon.$$

Whence we find for $\|x\| < \delta$ that

$$\begin{aligned} \|DP(x)\|_i &= \|DP(x) - DP(0) + DP(0)\|_i \\ &\leq \|DP(x) - DP(0)\|_i + \|DP(0)\|_i \leq \varepsilon. \end{aligned}$$

Combined with 8.1.4 in [46] (a generalization of the Mean Value Theorem to vector–valued functions), we find for all $x \in B_\delta(0)$,

$$\|P(x)\| \leq \sup_{s \in [0,1]} \|DP(sx)\|_i \|x\| \leq \varepsilon \|x\|.$$

Iterating, for all $k \in \mathbb{N}$ and $\|x\| < \delta$ we have $\left\|P^k(x)\right\| \leq \varepsilon^k \|x\|$. Since all norms on finite–dimensional vector spaces are equivalent, the desired result follows immediately. ■

**Remark 7.** *Let $(\Sigma, P)$ be an $n$–dimensional discrete–time dynamical system that satisfies the hypotheses of Lemma 4 near $\xi = P(\xi)$. Then $P$ has a coordinate representation $\widetilde{P}(z, \zeta) = (Az, N(z, \zeta))$ in a neighborhood of $\xi$ where $A$ is an invertible matrix, $N(0,0) = 0$, and $\operatorname{spec} D_\zeta N(0,0) = \{0\}$. Therefore given $\varepsilon > 0$ we can apply Lemma 5 to the nonlinearity $\widetilde{P}(z, \zeta) - (Az, 0) = (0, N(z, \zeta))$ to find $\delta, C > 0$ such that for all $(z, \zeta) \in B_\delta(0)$ and $k \in \mathbb{N}$:*

$$\left\|\widetilde{P}^k(z, \zeta) - \left(A^k z, 0\right)\right\| \leq C\varepsilon^k \|(z, \zeta)\|.$$

*We conclude that $P$ is arbitrarily well–approximated near $\xi$ by a diffeomorphism on a submanifold whose dimension equals $\operatorname{rank} DP^n(\xi)$.*

[6]The map need not be nilpotent simply because its linearization is; consider the map $P : \mathbb{R} \to \mathbb{R}$ defined by $P(x) = x^2$.

## Appendix B
## $C^1$ Linearization

The technical result we desire was originally established by Hartman in the course of proving that an invertible contraction is $C^1$–conjugate to its linearization[7]. The original statement in [45] only considered invertible contractions. However, as noted in [48], the proof in [45] of the result we require does not make use of invertibility and the conclusion is still valid if zero is an eigenvalue of the linearization. For details we refer the reader to [49], which also contains a generalization to *hyperbolic* periodic orbits whose eigenvalues satisfy genericity conditions.

**Theorem 4** (Induction Assertion in [45]). *Let $U \subset \mathbb{R}^n$ be a neighborhood of the origin and $P : U \to \mathbb{R}^n$ a $C^1$ map of the form*

$$P(x, y, z) = (Ax + X(x, y, z),\ By + Y(x, y, z),\ Cz)$$

*such that*

$$DP(0) = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

*where:*

1) $x \in \mathbb{R}^k$, $y \in \mathbb{R}^\ell$, $z \in \mathbb{R}^m$ *and* $k + \ell + m = n$*;*
2) $A \in \mathbb{R}^{k\times k}$, $B \in \mathbb{R}^{\ell\times\ell}$, *and* $C \in \mathbb{R}^{m\times m}$*;*
3) $X : \mathbb{R}^n \to \mathbb{R}^k$ *and* $Y : \mathbb{R}^n \to \mathbb{R}^\ell$ *are* $C^1$*;*
4) $D_xX$, $D_yX$, $D_xY$, *and* $D_yY$ *are uniformly Lipschitz continuous in* $(x, y)$*;*
5) $D_zX$ *and* $D_zY$ *are uniformly Hölder continuous in* $z$*;*

*Suppose all the eigenvalues of $B$ have the same magnitude, that the eigenvalues of $A$ have smaller magnitude and those of $C$ have larger magnitude than those of $B$, and all eigenvalues of $DP(0)$ lie inside the unit disc:*

$$\forall b, \beta \in \operatorname{spec} B : |b| = |\beta|\,;$$
$$\forall a \in \operatorname{spec} A, b \in \operatorname{spec} B, c \in \operatorname{spec} C : 0 \le |a| < |b| < |c| < 1.$$

*Then there is a neighborhood of the origin $V \subset \mathbb{R}^n$ and a $C^1$ diffeomorphism $\varphi : V \to \mathbb{R}^n$ of the form*

$$\varphi(x, y, z) = (x + \varphi_X(z),\ y + \varphi_Y(x, y, z),\ z)$$

*for which $D\varphi(0) = I$ and for all $(u, v, w) \in \varphi(V)$ we have*

$$(\varphi \circ P \circ \varphi^{-1})(u, v, w) = (Au + U(u, v, w),\ Bv,\ Cw)$$

*where:*

1) $U : \varphi(V) \to \mathbb{R}^k$ *is* $C^1$*;*
2) $D_uU$ *is uniformly Lipschitz continuous in* $(u, v, w)$*;*
3) $D_vU$ *and* $D_wU$ *are uniformly Lipschitz continuous in* $u$*;*
4) $D_vU$ *and* $D_wU$ *are uniformly Hölder continuous in* $(v, w)$.

**Remark 8.** *Theorem 4 may be applied inductively to exactly linearize all eigendirections corresponding to non–zero eigenvalues via a $C^1$ change–of–coordinates; this is the content of Lemma 4 in Section A-B2.*

[7]Readers may be more familiar with the Hartman–Grobman Theorem (see Theorem 1.4.1 in [31] or Theorem 7.8 in [47]) which states that the phase portrait near an exponentially stable fixed point of a discrete–time dynamical system is *topologically* conjugate to its linearization.

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