Numerical Integration of Hybrid Dynamical Systems via Domain Relaxation

Sam Burden, Humberto Gonzalez, Ram Vasudevan, Ruzena Bajcsy, and S. Shankar Sastry

***Abstract*— Though hybrid dynamical systems are a powerful modeling tool, it has proven difficult to accurately simulate their trajectories. In this paper, we develop a provably convergent numerical integration scheme for approximating trajectories of hybrid dynamical systems. This is accomplished by first relaxing hybrid systems whose continuous states reside on manifolds by attaching epsilon-sized strips to portions of the boundary and then extending the dynamic and distance metric onto these strips. On this space we develop a numerical integration scheme and prove that discrete approximations converge to trajectories of the hybrid system. An example is included to illustrate the approach.**

I. Introduction

Hybrid dynamical systems provide natural models for systems whose dynamics involve both continuous and discrete transitions. Critical to the study of such systems is numerical simulation. Two approaches to numerical simulation have been considered in the hybrid systems literature. The first method, *event detection*, aims to approximate the instant in time when a trajectory crosses a switching surface by constructing a polynomial approximation to the trajectory and then employing a root-finding scheme [5], [6], [16]. Unfortunately no proof exists that the approximation generated using this method converges to the actual trajectory. The second method, *time stepping*, uses a variable-step integrator to place events at sample times of the discrete approximation [3]. Convergence results exist for this method but only for the particular case of mechanical systems with impact [12], [13].

In this paper, we present a numerical integration algorithm to simulate hybrid dynamical systems whose continuous states evolve on smooth manifolds. First, we relax switching surfaces by attaching an epsilon-sized strip in a manner similar to the technique involved in regularizing Zeno executions [7]. We then extend the vector field and distance metric from each domain onto these strips to obtain a *relaxed hybrid dynamical system*. In a manner similar to the construction of the *hybrifold* [17] and *hybrid colimit* [2], we identify subsets of the relaxed domains to construct a single metric space and develop our numerical integration scheme on this space. Importantly, we prove that the discrete approximation generated by our algorithm converges to the original trajectory in this space.

Our contributions are twofold: first, in Section III we construct a metric space which contains the domains of a hybrid system and supports convergence analysis; second, in Section IV, we develop a discrete approximation technique and prove that this approximation converges to the original trajectory. Section II describes the notation used throughout the paper and Section V contains an example illustrating the numerical integration scheme.

The authors are with the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, Berkeley, CA 94720. {sburden,hgonzale,ramv,bajcsy,sastry}@eecs.berkeley.edu

II. Preliminaries

We begin by introducing the standard mathematical objects used throughout this paper. An extended introduction to the ideas presented herein can be found in [9]. A *topological n-dimensional manifold* is a set M that is locally equivalent to a subset of $\mathbb{R}^n$, i.e. there exists a collection of functions defined from a subset of M to $\mathbb{R}^n$, $\{\varphi_\alpha\}_\alpha$, such that for each point in $p \in M$ there exists a neighborhood of that point U_p with $\varphi_\alpha|_{U_p}$ a homeomorphism. The collection of functions $\{\varphi_\alpha\}_\alpha$ together with their respective domains are called the *charts* of M. A *smooth n-dimensional manifold* is a topological manifold where for each α and β, $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a diffeomorphism on its domain. A *manifold with boundary* is a manifold where the range of the charts is not $\mathbb{R}^n$, but $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, and its boundary ∂M corresponds to the union of the preimages of all charts of the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$. An *embedded k-dimensional submanifold* is a subset $S \subset M$ for which every $p \in S$ is contained in a chart φ_p over a domain U_p for which $\varphi_p(U_p \cap S) \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$; $(n-k)$ is the *codimension* of S. For instance, the boundary ∂M is an embedded $(n-1)$-dimensional submanifold without boundary.

Given a manifold M, $C^\infty(M)$ is the set of all the smooth functions from M to $\mathbb{R}$. The *tangent space* at a point $p \in M$, T_pM, is the set of all directional derivatives of smooth functions evaluated at p, i.e. given $V \in T_pM$ and $f \in C^\infty(M)$, then $V(f)$ is a directional derivative of f evaluated at p. The *tangent bundle* of M, TM, is the disjoint union of all the tangent spaces, i.e. $TM = \coprod_{p \in M} T_pM$, and a *vector field* is a map $V : M \to TM$ such that for each $p \in M$, $V(p) \in T_pM$. Also, given two manifolds M and N, and a smooth function $H : M \to N$, the *pushforward* $H_*|_p : T_pM \to T_{H(p)}N$ is defined as $\big(H_*|_p(V)\big)(f) = V(f \circ H)$. In practice, the pushforward can be understood as the Jacobian matrix of H evaluated at p, taking vectors from T_pM to $T_{H(p)}N$.

Given a smooth vector field V and a point p, an *integral curve of V at p* denoted $x : I \to M$, where $I \subset \mathbb{R}$ is a connected set containing the origin, is a curve satisfying

$$\dot{x}(t) = V\big(x(t)\big), \quad x(0) = p, \quad \forall t \in I. \tag{1}$$

We say $x : I \to M$ is a *maximal integral curve of V at p* if for any other integral curve $\tilde{x} : \widetilde{I} \to M$ of V at p, $\widetilde{I} \subset I$.

A *Riemannian metric at* $p \in M$ is a smooth bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$ such that $g_p(V,W) = g_p(W,V)$ for all $V, W \in T_pM$, and $g_p(V,V) > 0$ whenever $V \neq 0$. A *Riemannian metric on* M is a collection of Riemannian metrics at each point in M forming a smooth map $g : TM \times TM \to \mathbb{R}$. We can define an *induced distance* $d : M \times M \to [0,\infty)$ induced by the Riemannian metric to be the infimum of the length of piecewise smooth curves between the arguments of d. Given this definition, it can be shown that d generates the topology of M (see Lemma 6.2 in [8]).

A *retraction at* $p \in M$ is a continuous map $\beta_p : W_p \to M$, where $W_p \subset T_pM$ is a connected neighborhood of the origin, that is differentiable at the origin, with $\beta_p(0) = p$ and $(\beta_p)_*|_0 \equiv \mathrm{id}_{T_pM}$, where we use the canonical identification $T_0(T_pM) \simeq T_pM$ and id_{T_pM} is the identity function from T_pM to itself. A *retraction on* M is a collection of retractions at each point in M forming a map $\beta : W \to M$, where $W \subset TM$ is the disjoint union of all sets W_p. To appreciate the utility of the retraction, remember that on a manifold adding a point to a tangent vector defined at that point may not produce a new point on the manifold. The retraction provides a means to resolve this problem. An illustration of how retractions look on various manifolds is described in [1]. In particular, note that on $\mathbb{R}^n$ a trivial retraction is $\beta_p(v) = p + v$.

Now we can define the class of hybrid dynamical systems of interest in this paper.

Definition 1: *A* hybrid dynamical system *is a tuple* $\mathcal{H} = (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{R})$*, where:*

- $\mathcal{J}$ *is a finite set indexing the discrete states of* $\mathcal{H}$*;*
- $\Gamma \subset \mathcal{J} \times \mathcal{J}$ *is the set of edges, forming a directed graph structure over* $\mathcal{J}$*;*
- $\mathcal{D} = \{D_j\}_{j\in\mathcal{J}}$ *is the set of domains, where each* D_j *is a compact connected smooth* n_j*-dimensional Riemannian manifold with boundary, with Riemannian metric* g_j *and induced distance* d_j*;*
- $\mathcal{B} = \{\beta_j\}_{j\in\mathcal{J}}$ *is the set of retractions, where* β_j *is a retraction defined on* D_j*;*
- $\mathcal{F} = \{F_j\}_{j\in\mathcal{J}}$ *is the set of vector fields, where each* F_j *is a vector field defined on* D_j*;*
- $\mathcal{G} = \{G_e\}_{e\in\Gamma}$ *is the set of guards, where* $G_{(j,j')} \subset \partial D_j$ *is a guard in mode* $j \in \mathcal{J}$ *which defines a transition to mode* $j' \in \mathcal{J}$*;*
- $\mathcal{R} = \{R_e\}_{e\in\Gamma}$ *is the set of reset maps, where* $R_{(j,j')} : G_{(j,j')} \to \partial D_{j'}$ *is a continuous map.*

We make the following assumptions on the vector fields, guards, reset maps, and retractions:

Assumption 1: *Each vector field* F_j*,* $j \in \mathcal{J}$*, is Lipschitz continuous in each chart, i.e. for each chart* φ *of* D_j*,* $\widetilde{F}_j : \widetilde{U} \to T\mathbb{R}^{n_j}$ *defined as* $\widetilde{F}_j(p) = \varphi_*\left(F_j\left(\varphi^{-1}(p)\right)\right)$ *is Lipschitz where* $\widetilde{U} \subset \mathbb{R}^{n_j}$*.*

Assumption 2: *The guards do not intersect, i.e. for each pair* $e_1, e_2 \in \Gamma$*,* $e_1 \neq e_2$*,* $G_{e_1} \cap G_{e_2} = \emptyset$*.*

Assumption 3: *The guards are closed embedded submanifolds with codimension 1. Also, the image of each reset map is a closed set.*

Assumption 4: *There exists an open neighborhood of the origin that is contained by the domain of all of the retractions. Moreover the pushforward of each retraction in each chart is Lipschitz with respect to its point of evaluation, i.e. for each chart* φ *on* D_j*,* $\widetilde{\beta}_p : \widetilde{W} \to \mathbb{R}^{n_j}$ *defined as* $\widetilde{\beta}_p(V) = \varphi\left((\beta_j)_p\left(\varphi_*^{-1}(V)\right)\right)$ *where* $\widetilde{W} \subset T_p\mathbb{R}^{n_j}$ *has a pushforward* $(\widetilde{\beta}_p)_*|_V : T_V\widetilde{W} \to T_{\widetilde{\beta}_p(V)}\mathbb{R}^{n_j}$ *that is Lipschitz with respect to its point of evaluation* V*.*

Assumptions 1 and 2 are sufficient to ensure the existence and uniqueness of executions of hybrid dynamical systems as we prove in Proposition 3. Assumption 3 allows us to metrize our relaxed domain in Section III. Assumption 4 is critical in ensuring the well-posedness of the numerical integration scheme described in Section IV.

III. Relaxation of a Hybrid Dynamical System

Rather than approximating the time instant when a trajectory intersects a guard, we prove convergence of the numerical integration scheme described in the next section by relaxing the hybrid dynamical system. First, we relax hybrid domains along their guards and extend the definition of the domain's metric, vector field, and retraction onto this relaxation. Next, we attach the disparate domains to each other via a topological quotient and construct a single metric space in which we can prove convergence. Given a topological space $\mathcal{S}$ and a function $f : A \to B$, $A, B \subset \mathcal{S}$, we define the following equivalence relation:

$$A \sim B = \left\{(a,b) \in \mathcal{S} \times \mathcal{S} \mid a \in f^{-1}(b)\right\}. \tag{2}$$

We denote the quotient of $\mathcal{S}$ under $A \sim B$ by $\frac{\mathcal{S}}{A\sim B}$.

We begin by relaxing a single domain of a hybrid system, D_j. The relaxation is accomplished by first "stretching" the boundary of D_j by attaching an ε-sized strip along each guard. As we show in the next section, this eliminates the need to accurately detect guard satisfaction.

Definition 2: *The* relaxation of a hybrid domain D_j *of a hybrid dynamical system* $\mathcal{H}$ *is:*

$$D_j^\varepsilon = \frac{D_j \coprod \left(\coprod_{\{j' \mid (j,j')\in\Gamma\}} \left(G_{(j,j')} \times [0,\varepsilon]\right)\right)}{G_{(j,j')} \sim \left(G_{(j,j')} \times \{0\}\right)}. \tag{3}$$

We denote the strip *that is glued along each guard by* $S^\varepsilon_{(j,j')} = \left(G_{(j,j')} \times [0,\varepsilon]\right)$.

The equivalence relation in (3) is generated by the natural bijection between $G_{(j,j')}$ and $G_{(j,j')} \times \{0\}$.

Since $G_{(j,j')}$ is a closed embedded submanifold of ∂D_j, $S^\varepsilon_{(j,j')}$ is a compact smooth manifold. Each point in the strip has a representation as a pair $(\zeta, \tau) \in G_e \times [0,\varepsilon]$, and we refer to τ as the *transverse coordinate*. We can construct coordinate charts for relaxations by extending the existing coordinate charts in our original space.

Definition 3: *Let* φ *be a chart on* D_j *with domain* U*. The* relaxation of φ *is:*

$$U^\varepsilon = \frac{U \coprod \left(\partial U \times [0,\varepsilon]\right)}{\partial U \sim \left(\partial U \times \{0\}\right)}, \tag{4}$$

$$\varphi^\varepsilon(x) = \begin{cases} \varphi(x), & x \in U, \\ \big(\varphi(\zeta), \tau\big), & x = (\zeta, \tau) \in (\partial U \times [0, \varepsilon]). \end{cases} \tag{5}$$

Note that $\varphi^\varepsilon|_{U^\varepsilon}$ is a homeomorphism, and the equivalence relation in (4) is generated by the natural bijection between ∂U and $\partial U \times \{0\}$.

Next, we develop a metric on each relaxed domain. We aim to endow each relaxed domain D_j^ε with a metric $d_j^\varepsilon : D_j^\varepsilon \times D_j^\varepsilon \to [0, \infty)$ which restricts to d_j on D_j. To achieve this, we first define a metric on each strip and then prove that the metric induced by the quotient structure of the relaxation is actually a metric on D_j^ε with the desired property.

Definition 4: *The metric $d^\varepsilon_{(j,j')} : S^\varepsilon_{(j,j')} \times S^\varepsilon_{(j,j')} \to [0, \infty)$ on the strip $S^\varepsilon_{(j,j')}$ is:*

$$d^\varepsilon_{(j,j')}\big((\zeta, \tau), (\zeta', \tau')\big) = d_j\big(\zeta, \zeta'\big) + |\tau - \tau'|. \tag{6}$$

Using this definition and d_j, one can regard the disjoint union of D_j and all of the strips as a metric space using the disjoint union metric[1], denoted $\tilde{d}_j^\varepsilon$. Then the function $d_j^\varepsilon : D_j^\varepsilon \times D_j^\varepsilon \to [0, \infty)$ defined by:

$$d_j^\varepsilon(x, y) = \inf_{k \in \mathbb{N}} \left\{ \sum_{i=1}^{k} \tilde{d}_j^\varepsilon(p_i, q_i) \mid x = p_1, y = q_k, q_i \sim p_{i+1} \right\}$$

is a semi-metric on D_j^ε, i.e. it is non-negative, symmetric, and satisfies the triangle inequality (see Definition 3.1.12 in [4]). In general, $d_j^\varepsilon(x, y) = 0$ may not imply $x \sim y$. The following proposition establishes the fact that d_j^ε is a metric on D_j^ε.

Proposition 1: *The function d_j^ε is a metric on D_j^ε.*

Proof. Let $\pi_j : D_j \coprod \big(\coprod_{\{j' | (j,j') \in \Gamma\}} S^\varepsilon_{(j,j')} \big) \to D_j^\varepsilon$ denote the canonical quotient map sending each point to its equivalence class, i.e. $\pi(x) = [x]$. We already know that d_j^ε is a semi-metric, so all we must show is that $\pi_j(x) = \pi_j(y)$ whenever $d_j^\varepsilon\big(\pi_j(x), \pi_j(y)\big) = 0$.

Each $x \in D_j \setminus G_{(j,j')}$ has a d_j-ball that is disjoint from $G_{(j,j')}$, since $G_{(j,j')}$ is closed by Assumption 3, therefore $\pi^{-1}([x]) = \{x\}$. Similarly each $x \in \big(S^\varepsilon_{(j,j')} \setminus (G_{(j,j')} \times \{0\})\big)$ has a $d^\varepsilon_{(j,j')}$-ball which is disjoint from $G_{(j,j')} \times \{0\}$, therefore $\pi^{-1}([x]) = \{x\}$. Finally, each $x \in G_{(j,j')}$ has d_j-ball and $d^\varepsilon_{(j,j')}$-ball (defined in their appropriate space) disjoint from any other $y \in G_{(j,j')}$, therefore $\pi^{-1}([x]) = \{x, (x, 0)\}$. This argument is true for all $(j, j') \in \Gamma$, and thus establishes that d_j^ε is a metric on D_j^ε. □

Next, we extend the vector field onto the strip:

Definition 5: *For each $e \in \Gamma$, let the vector field on the strip S_e^ε, denoted F_e^ε, be the unit vector pointing outward along the transverse direction, i.e. $F_e^\varepsilon(\zeta, \tau) = \frac{\partial}{\partial \tau}$. Then the* relaxation of the vector field F_j *is:*

$$F_j^\varepsilon(x) = \begin{cases} F_j(x), & x \in D_j, \\ F^\varepsilon_{(j,j')}(x), & x \in G_{(j.j')} \times (0, \varepsilon], \ \forall (j, j') \in \Gamma. \end{cases}$$

Note that the relaxation of the vector field is generally not continuous along each $G_{(j,j')}$.

[1]Points in the same component of the disjoint union use the distance metric of the component, otherwise the distance is defined to be infinite.

In contrast to the vector field which we explicitly extend throughout the strip, we do not require an explicit form for our relaxed retraction. Instead we require that any relaxed retraction centered at points sufficiently close to a guard has a range that includes the strip.

Definition 6: *Let $p \in D_j \setminus \partial D_j$ and β_p be a retraction on $p \in D_j$, with domain W_p. If $\beta_p^{-1}\big(G_{(j,j')}\big) \cap W_p \neq \emptyset$, then a* relaxation of β_p *is any continuously differentiable function $\beta_p^\varepsilon : U_p \to D_j^\varepsilon$, with U_p an open set containing W_p, so that β_p^ε agrees with β_p on W_p. The* relaxation of a retraction β on D_j*, denoted by β^ε, is just the collection of relaxations of β_p^ε on the interior of D_j.*

Though we do not prove it here due to lack of space, such a relaxation of a retraction is always possible. Note in particular that if the domain is a subset of $\mathbb{R}^n$ then a relaxation of a retraction could be constructed by simply extending the domain of β_p and setting $\beta_p^\varepsilon(v) = p + v$.

We simultaneously relax each hybrid domain to define the relaxation of a hybrid dynamical system and then attach the disparate domains of the relaxed system together to construct a metric space.

Definition 7: *The* relaxation of a hybrid dynamical system $\mathcal{H}$ *is a tuple $\mathcal{H}^\varepsilon = (\mathcal{J}, \Gamma, \mathcal{D}^\varepsilon, \mathcal{B}^\varepsilon, \mathcal{F}^\varepsilon, \mathcal{G}^\varepsilon, \mathcal{R}^\varepsilon)$ where:*

- *$\mathcal{D}^\varepsilon = \{D_j^\varepsilon\}_{j \in \mathcal{J}}$ is the set of relaxations of each of the domains D_j, and $(D_j^\varepsilon, d_j^\varepsilon)$ is a metric space;*
- *$\mathcal{B}^\varepsilon = \{\beta_j^\varepsilon\}_{j \in \mathcal{J}}$ is the set of relaxations of each of the retractions β_j;*
- *$\mathcal{F}^\varepsilon = \{F_j^\varepsilon\}_{j \in \mathcal{J}}$ is the set of relaxations of each of the vector fields F_j;*
- *$\mathcal{G}^\varepsilon = \{G_e^\varepsilon\}_{e \in \Gamma}$ is the set of relaxations of guards in $\mathcal{G}$, where each guard $G_{(j,j')} \subset \partial D_j$ is relaxed to $G^\varepsilon_{(j,j')} = \big(G_{(j,j')} \times \{\varepsilon\}\big) \subset \partial D_j^\varepsilon$;*
- *$\mathcal{R}^\varepsilon = \{R_e^\varepsilon\}_{e \in \Gamma}$ is the set of relaxations of reset maps, where $R^\varepsilon_{(j,j')} : G^\varepsilon_{(j,j')} \to \partial D_{j'}$ is defined by $R^\varepsilon_{(j,j')}(\zeta, \tau) = R_{(j,j')}(\zeta)$.*

Definition 8: *The* relaxed hybrid quotient space *of a relaxed hybrid dynamical system $\mathcal{H}^\varepsilon$ is:*

$$\mathcal{M}^\varepsilon = \frac{\coprod_{j \in \mathcal{J}} D_j^\varepsilon}{G_e^\varepsilon \sim R_e^\varepsilon\big(G_e^\varepsilon\big)}. \tag{7}$$

Figure 1 illustrates this construction. The quotient in (7) is generated by each of the relaxed reset maps $\{R_e^\varepsilon\}_{e \in \Gamma}$.

As before, we may regard the disjoint union of the relaxed domains as a metric space with the disjoint union metric, $\tilde{\mu}^\varepsilon$ and use this to construct a metric on the quotient.

Proposition 2: *The function $\mu^\varepsilon : \mathcal{M}^\varepsilon \times \mathcal{M}^\varepsilon \to [0, \infty)$ defined by:*

$$\mu^\varepsilon(x, y) = \inf_{k \in \mathbb{N}} \left\{ \sum_{i=1}^{k} \tilde{\mu}^\varepsilon(p_i, q_i) \mid x = p_1, y = q_k, q_i \sim p_{i+1} \right\}$$

is a metric on $\mathcal{M}^\varepsilon$.

Proof. Since the image of each reset map is closed by Assumption 3 and since we can obtain metric neighborhoods separating distinct points in $R_e^\varepsilon\big(G_e^\varepsilon\big)$ for each $e \in \Gamma$, the

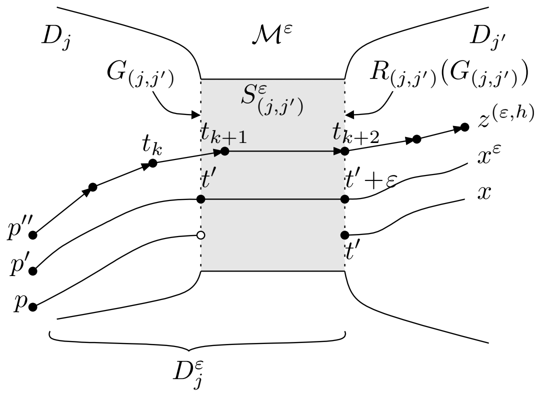


Fig. 1. Illustration of the relaxed quotient space $\mathcal{M}^\varepsilon$ constructed in Section III, as well as the execution x starting at p, a relaxed execution x^ε starting at p', and a discrete approximation $z^{(\varepsilon,h)}$ starting at p'', as defined in Section IV.

result follows by a similar argument to the one presented in the proof of Proposition 1. □

Observe that for all $x, y \in D_j$ and all $j \in \mathcal{J}$, $\mu^\varepsilon(x,y) \leq d^\varepsilon_j(x,y)$. Further, μ^ε has the useful property that if $y = R^\varepsilon_e(x)$ for some $x \in G^\varepsilon_e$ and $e \in \Gamma$, then $\mu^\varepsilon(x,y) = 0$. Exploiting this property, we define a metric between curves on $\mathcal{M}^\varepsilon$.

Definition 9: *With $I \subset [0,\infty)$ a bounded interval, given any two curves $x, x' : I \to \mathcal{M}^\varepsilon$ we define*

$$\rho^\varepsilon_I\big(x,x'\big) = \sup\big\{\mu^\varepsilon\big(x(t),x'(t)\big) \mid t \in I\big\}. \tag{8}$$

IV. Relaxed Executions and Discrete Approximations

This section contains the main result of this paper: discrete approximations to trajectories of hybrid dynamical systems, constructed using a modified version of the Forward Euler Algorithm, converge to the actual trajectories. First, we define executions of hybrid dynamical systems and relaxed hybrid dynamical systems. Next, we define our discrete approximation scheme on our relaxed space. Finally, we prove that the discrete approximations of executions of the relaxed hybrid dynamical system converge to the inclusion of the executions of the original hybrid dynamical system in our relaxed space.

We begin by defining a *trajectory* or *execution* of a hybrid dynamical system. This definition agrees with the standard intuition about executions of hybrid systems, i.e. the execution evolves as a standard dynamical system until a guard is reached, in which case a "jump" occurs via a reset map to a new hybrid domain. Since each domain in $\mathcal{D}$ is compact and each vector field in $\mathcal{F}$ is Lipschitz continuous by Assumption 1, every maximal integral curve of the vector field F_j through a point $p \in D_j$ either stops in finite time at a point $q \in \partial D_j$ or it continues in D_j indefinitely.

Definition 10: *An execution of a hybrid dynamical system $\mathcal{H}$ starting at a point $p \in D_j$, denoted by x, is defined by the following algorithm:*

1. *Set $x(0) = p$, $t = 0$, and set j to be the current mode indexing the domain of p.*
2. *Compute the maximal integral curve of F_j at $x(t)$, denoted by $\gamma : J \to D_j$, and set $x(t+s) = \gamma(s)$ for all $s \in J \cap [0,\infty)$. Set $t' = t + \sup\{s \mid s \in J\}$. Note that if t' is finite, then $x(t') \in \partial D_j$.*
3. *If $t' = \infty$ or there is no $G_{(j,j')} \in \mathcal{G}$ such that $x^\varepsilon(t') \in G_{(j,j')}$, then the execution stops.*
4. *Let $G_{(j,j')} \in \mathcal{G}$ such that $x(t') \in G_{(j,j')}$. Replace the value of $x(t')$ with $R_{(j,j')}\big(x(t')\big)$.*
5. *Set $t = t'$ and $j = j'$. Go to step 2.*

Hence x is a piecewise continuous function defined from an interval $I \subset [0,\infty)$ to $\coprod_{j\in\mathcal{J}} D_j$.

Note that any execution $x : I \to \coprod_{j\in\mathcal{J}} D_j$ can be regarded as a function $x : I \to \mathcal{M}^\varepsilon$ via the map $\coprod_{j\in\mathcal{J}} D_j \hookrightarrow \mathcal{M}^\varepsilon$ where $\hookrightarrow$ denotes the natural inclusion that sends each element of $\coprod_{j\in\mathcal{J}} D_j$ to its equivalence class in $\mathcal{M}^\varepsilon$.

Definition 11: *A* Zeno execution *is an execution where there exists an infinite number of discrete transitions in a finite amount of time. Hence, there exists $T \in \mathbb{R}$ so that the execution is only defined on $I = [0,T)$.*

Proposition 3: *The algorithm in Definition 10 yields a unique well-defined maximal trajectory starting at every point $p \in D_j$, for each $j \in \mathcal{J}$.*

Proof. First note that steps 2 and 4 of the algorithm result in a unique value for the execution x at a given time. Indeed, the integral of a curve is unique by Assumption 1, and if there exists a guard at a point in the boundary of a domain, then this guard is unique by Assumption 2. Hence, either the execution continues for all $t > 0$ as the integral curve of a vector field, or it stops at a finite time because the integral curve escapes the manifold or because the execution is Zeno. In either case we obtain a maximal trajectory, since its time domain cannot be extended. □

Using a similar algorithm we can define the execution of a relaxed hybrid dynamical system. We only define the algorithm starting from points in D_j since the strips are artificial objects which do not appear in the original system.

Definition 12: *An execution of a relaxed hybrid dynamical system $\mathcal{H}^\varepsilon$ starting at a point $p \in D_j$, denoted by x^ε, is defined by the following algorithm:*

1. *Set $x^\varepsilon(0) = p$, $t = 0$, and set j to be the current mode indexing the domain of p.*
2. *Compute the maximal integral curve of F_j at $x^\varepsilon(t)$, denoted by $\gamma : J \to D_j$, and set $x^\varepsilon(t+s) = \gamma(s)$ for all $s \in J \cap [0,\infty)$. Set $t' = t + \sup\{s \mid s \in J\}$. Note that if t' is finite, then $x^\varepsilon(t') \in \partial D_j$.*
3. *If $t' = \infty$ or there is no $G^\varepsilon_{(j,j')} \in \mathcal{G}$ such that $x^\varepsilon(t') \in G_{(j,j')}$, then the execution stops.*
4. *Let $G_{(j,j')} \in \mathcal{G}$ such that $x^\varepsilon(t') \in G_{(j,j')}$, then compute the maximal integral curve of $F^\varepsilon_{(j,j')}$ at $x^\varepsilon(t')$, denoted by $\gamma' : [0,\varepsilon] \to S^\varepsilon_{(j,j')}$, set $x^\varepsilon(t'+s) = \gamma'(s)$ for all $s \in [0,\varepsilon)$, and set $x^\varepsilon(t'+\varepsilon) = R_{(j,j')}\big(\gamma'(\varepsilon)\big)$.*
5. *Set $t = t' + \varepsilon$ and $j = j'$. Go to step 2.*

Hence x^ε is a continuous function defined from an interval $I \subset [0,\infty)$ into $\mathcal{M}^\varepsilon$.

Note that the definition of a relaxed execution is in practice only a delayed version of the original definition of an execution, where the relaxed version has to spend additional time on each relaxed guard before making its transition. Also, note that the definition above is comparable to the execution of a regularized hybrid system in [7].

Next, we define the types of trajectories that can be approximated.

Definition 13: *Let x_p^ε be the relaxed execution starting at p. We say an execution $x_{p'}^\varepsilon : I \to \mathcal{M}^\varepsilon$ of a relaxed hybrid dynamical system $\mathcal{H}^\varepsilon$ is* orbitally stable *at $p' \in \mathcal{M}^\varepsilon$ if, for each $t \in I$, the map $p \mapsto x_p^\varepsilon(t)$ is continuous at p'.*

Orbitally stable executions are exactly the type of execution that can be approximated in a hybrid dynamical system [10]. Indeed, if an execution is not orbitally stable then there exists a time t' and an execution which when initialized arbitrarily close to $x^\varepsilon(t')$ yields a trajectory with a different sequence of discrete transitions. We state the following proposition without proof, but it follows directly from Proposition 3.

Proposition 4: *The algorithm in Definition 12 yields a unique well-defined maximal trajectory starting at every point $p \in D_j$, for each $j \in \mathcal{J}$.*

We introduce the following assumption to rule out pathological Zeno executions.

Assumption 5: *Let $x : [0, T) \to \mathcal{M}^\varepsilon$ be a Zeno execution, then there exists a finite set $\{p_k\}_{k=1}^m \subset \mathcal{M}^\varepsilon$ such that for each $r > 0$ there exists $\delta > 0$ so that for each $t \in [T - \delta, T)$ there exists $k \in \{1, \ldots, m\}$ satisfying $\mu^\varepsilon\big(p_k, x(t)\big) \leq \varepsilon + r$. Moreover, for each pair $i, j \in \{1, \ldots, m\}$, $\mu^\varepsilon(p_i, p_j) \leq m\varepsilon$.*

This assumption serves two purposes. First, it guarantees that Zeno executions accumulate in a finite collection of points, eliminating pathological cases where an infinite number of transitions cause an execution to “fill” a portion of the domain [18]. Second, it guarantees that each accumulation point is the image of another accumulation point through a reset map, enabling us to bound their distance in $\mathcal{M}^\varepsilon$ as a function of ε.

Theorem 1: *Let $\mathcal{H}^\varepsilon$ be the relaxation of the hybrid dynamical system $\mathcal{H}$, suppose $x : I \to \coprod_{j\in\mathcal{J}} D_j$ is an execution of $\mathcal{H}$ over the bounded interval $I \subset [0, \infty)$, and let x^ε be its relaxation. Then $\rho_I^\varepsilon\big(x, x^\varepsilon\big) \to 0$ as $\varepsilon \to 0$.*

Proof. First, let us assume that x undergoes a finite number of discrete transitions on I, thus $I = [0, T]$ for some $T > 0$. Let $\{t_k\}_{k=0}^{m+1}$ denote the sequence of transition times, where $t_0 = 0$ and $t_{m+1} = T$. Without loss of generality, consider ε sufficiently small such that $(m + 1)\varepsilon < t_{k+1} - t_k$ whenever $t_{k+1} > t_k$.

Since the function $x|_{[t_k,t_{k+1})}$ is defined on D_j for some $j \in \mathcal{J}$, using the definition of arc length we get that for each pair $t, t' \in [t_k, t_{k+1})$, $t' \geq t$,

$$\begin{aligned}\mu^\varepsilon\big(x(t), x(t')\big) &\leq d_j\big(x(t), x(t')\big) \\ &\leq \int_t^{t'} g_j(\dot{x}(s), \dot{x}(s))ds \leq L(t' - t),\end{aligned} \tag{9}$$

where $L = \sup\big\{\|F_j(p)\| \mid p \in D_j,\ j \in \mathcal{J}\big\}$.

Let $k \geq 1$, and assume that $t_{k+1} > t_k$. Then we compute a bound on the distance $\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big)$ for $t \in [t_k, t_{k+1})$, by considering three cases:

- $t \in [t_k, t_k + (k-1)\varepsilon)$: In this case x^ε is in the domain that x just left, thus $x^\varepsilon(t) = x(t - (k-1)\varepsilon)$. If we define $x(t_k^-) = \lim_{t\to t_k^-} x(t)$, then
$$\begin{aligned}\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) &\leq \mu^\varepsilon\big(x(t-(k-1)\varepsilon), x(t_k^-)\big) + \\ &\quad + \mu^\varepsilon\big(x(t_k^-), x(t_k)\big) + \\ &\quad + \mu^\varepsilon\big(x(t_k), x(t)\big) \\ &\leq L\,(k-1)\varepsilon + \varepsilon + L\,(k-1)\varepsilon.\end{aligned}$$
- $t \in [t_k + (k-1)\varepsilon, t_k + k\varepsilon)$: In this case x^ε is in the strip, hence
$$\begin{aligned}\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) &\leq \mu^\varepsilon\big(x^\varepsilon(t), x(t_k)\big) + \mu^\varepsilon\big(x(t_k), x(t)\big) \\ &\leq \varepsilon + L\,k\varepsilon.\end{aligned}$$
- $t \in [t_k + k\varepsilon, t_{k+1})$: In this case both x^ε and x are in the same domain, hence $x^\varepsilon(t) = x(t - k\varepsilon)$ and $\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) \leq L\,k\varepsilon$.

If there is a sequence of ℓ consecutive jumps such that $t_k = \ldots = t_{k+\ell-1}$, then the argument is analogous, noting that in the second case the interval of interest is $[t_k+(k-1)\varepsilon, t_k+(k+\ell)\varepsilon)$, with a bound $\ell\varepsilon+L(k+\ell)\varepsilon$. Also, note that $\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) = 0$ whenever $t \in [t_0, t_1)$, and $\mu^\varepsilon\big(x^\varepsilon(t_{m+1}), x(t_{m+1})\big) \leq \mu^\varepsilon\big(x^\varepsilon(t_{m+1}^-), x(t_{m+1}^-)\big) + \varepsilon$, this last bound being an equality if there is a discrete transition exactly at $t = t_{m+1}$. Therefore, it follows that

$$\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) \leq (2L + 1)m\varepsilon, \tag{10}$$

which proves the theorem if the number of transitions is finite.

Next, let us consider the case when there is an infinite number of transitions on I, thus x is a Zeno execution and $I = [0, T)$. Let $r > 0$, and note that as a consequence of Assumption 5 (and using its notation) there exists $\delta_{\frac{r}{2}} > 0$ such that $\mu^\varepsilon\big(p_1, x^\varepsilon(t)\big) \leq m\varepsilon + \frac{r}{2}$ for each $t \in [T - \delta_{\frac{r}{2}}, T)$. Let us denote by $m(\delta)$ the number of discrete transitions on the interval $[0, T - \delta]$, then choose $\varepsilon > 0$ such that

$$m(\delta_{\frac{r}{2}})\varepsilon < \min\big\{t_{m(\delta_{\frac{r}{2}})+1} - t_{m(\delta_{\frac{r}{2}})}, \delta_r\big\}, \tag{11}$$

where the sequence $\{t_k\}_{k=0}^{m(\delta_{\frac{r}{2}})+1}$ is defined by the times of the discrete transitions of x. From this choice of ε we get that both x^ε and x are in the same domain at $t = T - \delta(\frac{r}{2})$, hence $x^\varepsilon(t) = x\big(t - m\big(\delta_{\frac{r}{2}}\big)\varepsilon\big)$, and since $m(\delta_{\frac{r}{2}})\varepsilon < \delta_r$, then

$$\mu^\varepsilon\big(p_1, x^\varepsilon(t)\big) \leq n\varepsilon + r, \quad \forall t \in \big[T - \delta_{\frac{r}{2}}, T\big). \tag{12}$$

Finally, as $\varepsilon \to 0$, $\rho_{[0,T)}^\varepsilon\big(x^\varepsilon(t), x(t)\big) \leq r$. But $r > 0$ was arbitrary, hence $\rho_{[0,T)}^\varepsilon\big(x^\varepsilon(t), x(t)\big) \to 0$ as $\varepsilon \to 0$. $\square$

Now we can define the discrete approximation of a relaxed execution. As mentioned above, our discrete approximation is based on the Forward Euler integration approximation for ODE's, modified to be applicable on Riemannian manifolds.

Definition 14: *Let $\mathcal{H}^\varepsilon$ be a relaxed hybrid dynamical system, and let $p \in D_j$ for some $j \in \mathcal{J}$. We say that the* discrete approximation of x^ε with step-size h *is the function $z^{(\varepsilon,h)}$ defined as follows:*

(1) *Set $z^{(\varepsilon,h)}(0) = p$, $t_0 = 0$, and $k = 0$.*

(2) *If $\beta_j^\varepsilon\big(tF_j\big(z^{(\varepsilon,h)}(t_k)\big)\big)$ is undefined for all $t>0$, then stop.*

(3) *Find the smallest $n \in \mathbb{N}$ such that $\frac{h}{2^n}F_j\big(z^{(\varepsilon,h)}(t_k)\big)$ is in the domain of β_j^ε, set $t_{k+1} = t_k + \frac{h}{2^n}$, and for each $t \in \big[0, \frac{h}{2^n}\big]$,*

$$z^{(\varepsilon,h)}(t+t_k) = \beta_j^\varepsilon\left(tF_j\big(z^{(\varepsilon,h)}(t_k)\big)\right). \tag{13}$$

(4) *If $z^{(\varepsilon,h)}(t_{k+1}) \notin S^\varepsilon_{(j,j')}$ for some $j' \in \mathcal{J}$, set $k=k+1$ and go to step 2.*

(5) *Otherwise $z^{(\varepsilon,h)}(t_{k+1}) \in S^\varepsilon_{(j,j')}$ Let $\tau \in [0,\varepsilon]$ denote the transverse coordinate of $z^{(\varepsilon,h)}(t_{k+1})$. Set $t_{k+2} = t_{k+1} + \varepsilon - \tau$. Since the vector field $F^\varepsilon_{(j,j')}$ on the strip is trivial, let $z^{(\varepsilon,h)}|_{[t_{k+1},t_{k+2}]}$ be its integral curve and replace the value of $z^{(\varepsilon,h)}(t_{k+2})$ with $R^\varepsilon_{(j,j')}\big(z^{(\varepsilon,h)}(t_{k+2})\big)$. Set $k=k+2$, $j=j'$, and go to step 2.*

The function $z^{(\varepsilon,h)}$ maps an interval $I \subset [0,\infty)$ into $\mathcal{M}^\varepsilon$.

Note that by Assumption 4, there exists $n_0 \in \mathbb{N}$ such that the value of n in step 3 is always greater that n_0. Figure 1 illustrates the trajectories of interest.

Theorem 2: *Let $\mathcal{H}^\varepsilon$ be a relaxed hybrid dynamical system, let $p \in D_j$ for some $j \in \mathcal{J}$, and let $I=[0,T]$ for some $T>0$. If the relaxed execution x^ε starting at p with domain I is orbitally stable, then there exists $K_0>0$ such that $\rho_I^\varepsilon(x^\varepsilon, z^{(\varepsilon,h)}) \to K_0\varepsilon$ as $h \to 0$.*

Proof. We complete this proof in incremental steps, first showing the convergence on a single domain, and then on the relaxed quotient space $\mathcal{M}^\varepsilon$.

Let $x:[0,T]\to D_j$ be the maximal integral curve of F_j at p on the interval $[0,T]$, and $z^h:[0,T]\to M$ be the discrete time approximation of x constructed using Definition 14 (note that x^ε and x are equal in D_j). Let $\{t_k\}_{k=0}^\ell \subset [0,T]$ be the sequence of time samples derived from the algorithm in Definition 14, where $t_0=0$ and we remove the dependence of each t_k and ℓ on h for notational convenience. Given a chart φ, let us also define $\widetilde{\beta}: \widetilde{W} \to \mathbb{R}^n$, $\widetilde{W} \subset T\mathbb{R}^n$, and $\widetilde{F}: \widetilde{U} \to T\mathbb{R}^n$, $\widetilde{U} \subset \mathbb{R}^n$, such that

$$\widetilde{\beta}_p(V) = \varphi\left((\beta_j)_p\left(\varphi_*^{-1}(V)\right)\right), \tag{14}$$

$$\widetilde{F}(q) = \varphi_*\left(F_j\left(\varphi^{-1}(q)\right)\right), \tag{15}$$

and similarly denote $\tilde{x} = \varphi \circ x$ and $\tilde{z}^h = \varphi \circ z^h$.

Let us assume that given $i, i' \le \ell$, both functions $x|_{[t_i,t_{i'})}$ and $z^h|_{[t_i,t_{i'})}$ are in the same chart. Then, using Picard Lemma (see Lemma 5.6.3 in [14]), for all $t \in [t_i, t_{i'})$,

$$\|\tilde{x}(t)-\tilde{z}^h(t)\| \le e^{\widetilde{L}(t_{i'}-t_i)}\Bigg(\big\|\tilde{x}(t_i)-\tilde{z}^h(t_i)\big\| + \sum_{k=i}^{i'-1}\int_{t_k}^{t_{k+1}}\big\|\widetilde{F}_k - (\widetilde{\beta}_k)_*|_{(s-t_k)\widetilde{F}_k}\widetilde{F}_k\big\|ds\Bigg), \tag{16}$$

where $\widetilde{L}$ is the Lipschitz constant of $\widetilde{F}$, $\widetilde{F}_k = \widetilde{F}\big(\tilde{z}^h(t_k)\big)$, and $\widetilde{\beta}_k = \widetilde{\beta}_{\tilde{z}^h(t_k)}$. Now, if we let $L_{\widetilde{F}} = \sup\big\{\|\widetilde{F}(q)\| \mid q \in \widetilde{U}\big\}$ and $L_{\widetilde{\beta}_*}$ be the Lipschitz constant of $\widetilde{\beta}_*$, and since $\widetilde{\beta}_*|_0 = I_{n_j}$, then

$$\big\|\widetilde{F}_k - (\widetilde{\beta}_k)_*|_{(s-t_k)\widetilde{F}_k}\widetilde{F}_k\big\| \le L^2_{\widetilde{F}}L_{\widetilde{\beta}_*}(s-t_k), \tag{17}$$

and therefore, for all $t \in [t_i, t_{i'})$,

$$\|\tilde{x}(t)-\tilde{z}^h(t)\| \le e^{\widetilde{L}(t_{i'}-t_i)}\Big(\big\|\tilde{x}(t_i)-\tilde{z}^h(t_i)\big\| + \frac{1}{2}L^2_{\widetilde{F}}L_{\widetilde{\beta}_*}h\Big). \tag{18}$$

Since D_j is compact, there exists a finite set of charts $\{\varphi_i\}_{i=1}^\nu$ such that their domains $\{U_i\}_{i=1}^\nu$ form a cover of D_j. Now let r_0 be defined by:

$$r_0 = \inf_{i\in\{1,\ldots,\nu\}}\ \inf_{q\in\partial U_i}\ \sup_{j\neq i}\ \inf_{q'\in\partial U_j\cap U_i}\|q-q'\|, \tag{19}$$

and notice that $r_0>0$ since every point at a boundary of a domain is at a positive distance from another boundary since the boundaries are closed and there are only a finite number of them. Then, for each point $q \in U_i$, there exists a neighborhood of q with radius at least r_0 contained in some U_j, $j \neq i$.

Note that given a chart φ_i with domain $U_i \subset D_j$, if $q, q' \in U_i$ then, given $\alpha(t) = \varphi_i^{-1}\big((1-t)\varphi_i(q) + t\varphi_i(q')\big)$ for $t \in [0,1]$,

$$d_j(q,q') \le \int_0^1 g_j(\dot\alpha(t),\dot\alpha(t))dt \le C\|\varphi_i(q)-\varphi_i(q')\|, \tag{20}$$

where $C = \sup\big\{\|(\varphi_i)_*^{-1}|_p\| \mid p \in U_i,\ i \in \{1,\ldots,\nu\}\big\}$. For notational convenience, since all the constants in equation (18) depend on the chart chosen, let $K = \sup\big\{\frac{1}{2}L^2_{\widetilde{F}_i}L_{\widetilde{\beta}_{i*}} \mid i \in \{1,\ldots,\nu\}\big\}$ and $L = \sup_i \widetilde{L}_i$.

Now we can finish the argument for executions on a manifold. Suppose that $x(0)$ and $z^h(0)$ are in the same chart and that $\|\tilde{x}(0)-\tilde{z}^h(0)\| \le \delta$, then there exists $s>0$ such that $x|_{[0,s)}$ and $z^h|_{[0,s)}$ are in the same chart, thus $d_j\big(x(s),z^h(s)\big) \le e^{LT}CKh+\delta$. Without loss of generality assume that either $x(s)$ or $z^h(s)$ is at the boundary of a chart. If h and δ are small enough then this distance is smaller than $\frac{r_0}{2}$, and in that case there exists $s'>s$ such that, for an interval $[s,s')$, both functions are again in the same domain U_i. Also note that using an argument similar to Equation (9), we get that $s'-s \ge \frac{r_0}{2L_{\widetilde{F}_i}}$. Therefore, after repeating this process a finite number of times, say $N \in \mathbb{N}$, we get that, for all $t \in [0,T]$,

$$d\big(x(t),z^h(t)\big) \le e^{LT}NCK(\delta+h). \tag{21}$$

Let us consider the case of relaxed domains. Let $q \in D_j^\varepsilon$ be in the same chart φ as $p \in D_j$, and assume $\|\varphi(p)-\varphi(q)\| \le \delta$. Hence, if $z^{(\varepsilon,h)}$ is the discrete approximation starting at q, Equation (21) is satisfied for the distance between x^ε and $z^{(\varepsilon,h)}$ as long as they do not transition onto a strip. Suppose that there exists t' such that $x^\varepsilon(t') \in G_{(j,j')}$ for some $j' \in \mathcal{J}$. Since x^ε is assumed orbitally stable, there exists δ small enough such that $z^{(\varepsilon,h)}(s^h) \in G_{(j,j')}$ for some $s^h \in [0,T]$. Let $b \in \mathbb{N}$ be such that $s^h \in [t_b, t_{b+1})$, where the sequence $\{t_k\}_{k=1}^\ell$ is as in Definition 14, and note that we have removed the dependence of b on h for notational convenience. Since by definition x^ε crosses the guard at a unique point and so does $z^{(\varepsilon,h)}$, we know that $s^h \to t'$ as $h \to 0$, thus for all $\delta' > 0$ there exists h small enough such that $|t' - t_{b+1}| \leq \delta' + h$.

Let us define the following times:

$$\sigma = \min\{t_{b+1}, t'\}, \qquad \sigma' = \max\{t_{b+1}, t'\}, \tag{22}$$
$$\omega = \min\{t_{b+2}, t' + \varepsilon\}, \quad \omega' = \max\{t_{b+2}, t' + \varepsilon\}, \tag{23}$$

then on the interval $[0, \sigma)$ we can still use the bound in Equation (21). On the interval $[\sigma, \sigma')$ one execution has transitioned into a strip, while the other is still governed by the vector field on D_j. On the interval $[\sigma', \omega)$ both executions are inside the strip, and on the interval $[\omega, \omega')$ one execution has transitioned to a new domain, while the second is still on D_j^ε. After time ω' both executions are in a new domain, and we can repeat the process. Therefore, we need to find bounds for the distance between x^ε and $z^{(\varepsilon,h)}$ on each of these intervals.

Let $L_{\widetilde{F}^\varepsilon} = \sup\{\|\widetilde{F}_i^\varepsilon(q)\| \mid q \in \varphi_i(U_i),\ i \in \{1, \ldots, \nu\}\}$ and $L_{\widetilde{\beta}^\varepsilon} = \sup\{\|(\beta_p^\varepsilon)_*\| \mid p \in \varphi_i(U_i),\ i \in \{1, \ldots, \nu\}\}$, and note that these constants are bounded for bounded ε. Then using Equation (9) as we did in the proof of Theorem 1 and Equation (21) we get that

$$\mu^\varepsilon\big(x^\varepsilon(\sigma'), z^{(\varepsilon,h)}(\sigma')\big) \leq e^{LT} NCK(\delta + h) + \\ + L_{\widetilde{F}^\varepsilon}\big(1 + L_{\widetilde{\beta}^\varepsilon}\big)(\delta' + h), \tag{24}$$

also

$$\mu^\varepsilon\big(x^\varepsilon(\omega), z^{(\varepsilon,h)}(\omega)\big) \leq \mu^\varepsilon\big(x^\varepsilon(\sigma'), z^{(\varepsilon,h)}(\sigma')\big) + 2\varepsilon, \tag{25}$$

where we use the fact that $\big\|\widetilde{F}_e^\varepsilon\big\| = 1$ for all $e \in \Gamma$ and $\varepsilon > 0$, and

$$\mu^\varepsilon\big(x^\varepsilon(\omega'), z^{(\varepsilon,h)}(\omega')\big) \leq \mu^\varepsilon\big(x^\varepsilon(\omega), z^{(\varepsilon,h)}(\omega)\big) + \\ + L'_{\widetilde{F}^\varepsilon}\big(1 + L'_{\widetilde{\beta}^\varepsilon}\big)(\delta' + 2h), \tag{26}$$

because $|t' + \varepsilon - t_{b+2}| \leq \delta' + 2h$ by the construction in Definition 14, and where $L'_{\widetilde{F}^\varepsilon}$ and $L'_{\widetilde{\beta}^\varepsilon}$ are defined similar to their original counterparts, but using the charts in hybrid domain j'.

At this point the generalization to relaxed executions defined on $\mathcal{M}^\varepsilon$ and their discrete approximations follows by noting that they have the same initial condition, they perform a finite number of discrete jumps on any bounded interval, and the number of discrete modes is finite, and that δ' can be chosen arbitrarily small. With that information we can construct constants K_0 and K_1 such that, for all $t \in [0, T]$, $\mu^\varepsilon\big(x^\varepsilon(t), z^{(\varepsilon,h)}(t)\big) \leq K_0\varepsilon + K_1 h$, proving the theorem. □

Corollary 1: *Let $\mathcal{H}$ be a hybrid dynamical system and $\mathcal{H}^\varepsilon$ be its relaxation. Let $p \in D_j$ for some $j \in \mathcal{J}$, and let $I \subset [0, \infty)$ be any bounded interval, subset of the domain of execution x starting at p. If x^ε is orbitally stable at p, then $z^{(\varepsilon,h)}$, starting at p, converges to x as $\varepsilon, h \to 0$, i.e.*

$$\lim_{\substack{\varepsilon \to 0 \\ h \to 0}} \rho_I^\varepsilon\big(x, z^{(\varepsilon,h)}\big) = 0 \tag{27}$$

Proof. Note that, by Theorem 1, this corollary is equivalent to prove that $\rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big) \to 0$ as both $\varepsilon, h \to 0$. Hence we show that $\rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big)$ converges uniformly on h as $\varepsilon \to 0$. Together with the result in Theorem 2, this gives the desired result.

Using an argument similar to the one in the proof of Theorem 7.9 in [15], this corollary is equivalent to showing

$$\lim_{h \to 0} \limsup_{\varepsilon \to 0} \rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big) = 0 \tag{28}$$

but from the proof of Theorem 2 this is clearly true. □

V. Implementation and Examples

In this section, we describe the information required to implement the algorithm in Definition 14, and then present an example. There are two main issues to consider before implementing our numerical scheme. To simplify the exposition, we will consider the case where the hybrid system is comprised of a single domain D with a single guard G. There is no loss of generality in specializing the discussion to this particular case.

First, the algorithm needs a collection of charts whose domains form a cover of D, and a way to determine which charts contain a given point in the domain. Since D is compact, only a finite number of charts is required. Note that if D admits a single chart, as in the case of D being a subset of $\mathbb{R}^n$ where the chart is trivial, then the implementation is greatly simplified. Second, in some problems the boundary of the manifold is not described as the preimage of the set $\{(x_1, \ldots, x_n) \mid x_n = 0\}$ under a boundary chart, but rather as the zero section of a smooth function $\lambda : D \to \mathbb{R}$, i.e. $\partial D = \lambda^{-1}(0)$ and $\lambda_*(x) \neq 0$ for all $x \in \partial D$. In this case, since D is compact, and given ε sufficiently small, the value of λ can be used as the transverse coordinate on the strip $S^\varepsilon = G \times [0, \varepsilon]$, which can be used in place of the boundary charts.

We illustrate the implementation[2] of our numerical scheme to approximate trajectories of a double pendulum with a mechanical stop; see Figure 2a for a schematic. Prior to an impact with the mechanical stop (i.e. while the motion is *unconstrained*), the system has two angular degrees of freedom, $q = (\theta_1, \theta_2)$, and the dynamics are Lagrangian, i.e. they have the form

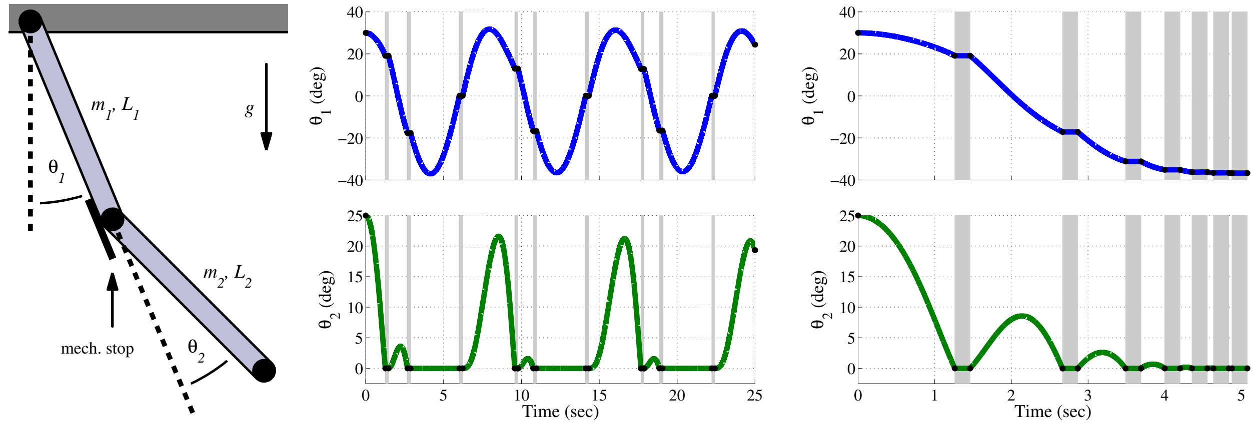
$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \partial_q V(q) = 0, \tag{29}$$

where $M(q)$ is the mass matrix, $C(q, \dot{q})$ is the Coriolis matrix, and $V(q)$ is the potential energy at configuration q. We refer the reader to [11] for the explicit expressions of these functions. When the second link collides with the mechanical stop (i.e. when $\theta_2 = 0$), the velocities are updated according to the impact law

$$(\dot{\theta}_1, \dot{\theta}_2) \mapsto \left(\dot{\theta}_1 - (1 + c)\dot{\theta}_2 \frac{\big(M(q)^{-1}\big)_{1,2}}{\big(M(q)^{-1}\big)_{2,2}}, -c\,\dot{\theta}_2\right), \tag{30}$$

where $c \in [0, 1]$ is the *coefficient of restitution* After impact, the system is re-initialized in a different discrete mode depending on the value of this constant. If $c > 0$, the system

[2]Code is available at `http://purl.org/sburden/cdc2011`



(a) schematic of double pendulum (b) trajectory with plastic impact, $c = 0$ (c) Zeno trajectory, $c = 0.5$

Fig. 2. (a) Planar double pendulum with mechanical stop; θ_1 gives the angle of the first link with respect to vertical, and θ_2 gives the angle of the second link with respect to the first. The link masses are m_1 and m_2, and their lengths are L_1 and L_2. A gravitational force points downward with constant g. (b) Trajectory of double pendulum with plastic impact, i.e. $c = 0$. (c) Zeno trajectory of double pendulum, $c = 0.5$. Both (b) and (c) start at initial condition $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (30°, 25°, 0, 0)$ and use Euler step-size $h = 0.001$ and strip size $\varepsilon = 0.2$. Vertical gray bars indicate when the simulation resides in the strip.

is reset to the *unconstrained* mode and simulation continues as before. If $c = 0$, the system enters the *constrained* state until the virtual force required to enforce the constraint $\theta_2 = 0$ becomes non-positive. The force is

$$\lambda(q, \dot{q}) = -\frac{\left(M(q)^{-1}\right)_{2,1:2}\left(C(q,\dot{q})\dot{q} + \partial_q V(q)\right)}{\left(M(q)^{-1}\right)_{2,2}}. \qquad (31)$$

It was shown in [11] that when $\theta_2 = 0$, either $\lambda > 0$ and $\ddot{\theta}_2 = 0$ (i.e. the constraint is maintained), or $\lambda = 0$ and $\ddot{\theta}_2 > 0$ (i.e. the system transitions to unconstrained motion), and thus the description of the system's dynamics is self-consistent.

An illustration of the execution with different values for the coefficient restitution are shown in Figures 2b and 2c. Observe that in either instance there is an epsilon sized delay due to the addition of the strip. In particular, notice that in the case of Zeno the strips begin to accumulate.

VI. Conclusion

In this paper we developed a provably convergent method to numerically approximate trajectories of hybrid dynamical systems. Rather than precisely detecting discrete switching events, we relax the domains of the hybrid system by "stretching" the guards by a small amount. Then we demonstrate that a Forward Euler scheme on the relaxed space produces approximations to trajectories of the original hybrid system. Our method can accommodate nonlinear dynamics and non-planar guards, and hence provides the strongest simulation result yet obtained for such systems. Further, the domain relaxation technique we develop may find use as a fundamental tool supporting analysis of hybrid systems.

References

[1] P. Absil, R. Mahony, and R. Sepulchre. *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, 2008.

[2] A. Ames and S. Sastry. A homology theory for hybrid systems: hybrid homology. In *Proc. of the 8th International Workshop on Hybrid Systems: Computation and Control*, pages 86–102, 2005.

[3] B. Brogliato, A. A. ten Dam, L. Paoli, F. Genot, and M. Abadie. Numerical simulation of finite dimensional multibody nonsmooth mechanical systems. *Applied Mechanics Reviews*, 55(2):107–150, 2002.

[4] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry*. American Mathematical Society, 2001.

[5] M. B. Carver. Efficient integration over discontinuities in ordinary differential equation simulations. *Mathematics and Computers in Simulation*, 20(3):190–196, 1978.

[6] J. Esposito, V. Kumar, and G. Pappas. Accurate event detection for simulating hybrid systems. In *Proc. of the 4th International Workshop on Hybrid Systems: Computation and Control*, pages 204–217, 2001.

[7] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry. On the regularization of zeno hybrid automata. *Systems & Control Letters*, 38(3):141–150, 1999.

[8] J. M. Lee. *Riemannian Manifolds: An Introduction to Curvature*. Springer, 1997.

[9] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, 2002.

[10] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48(1):2–17, 2003.

[11] Y. Or and A. D. Ames. Formal and practical completion of lagrangian hybrid systems. In *Proc. of the 2009 American Control Conference*, pages 3624–3631, 2009.

[12] L. Paoli and M. Schatzman. A numerical scheme for impact problems I: The one-dimensional case. *SIAM Journal on Numerical Analysis*, 40(2):702–733, 2003.

[13] L. Paoli and M. Schatzman. A numerical scheme for impact problems II: The multidimensional case. *SIAM Journal on Numerical Analysis*, 40(2):734–768, 2003.

[14] E. Polak. *Optimization: Algorithms and Consistent Approximations*. Springer, 1997.

[15] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1964.

[16] L. Shampine, I. Gladwell, and R. Brankin. Reliable solution of special event location problems for ODEs. *ACM Transactions on Mathematical Software*, 17(1):11–25, 1991.

[17] S. N. Simic, K. N. Johansson, J. Lygeros, and S. Sastry. Towards a geometric theory of hybrid systems. *Dynamics of Continuous, Discrete, and Impulsive Systems, Series B*, 12(5-6):649–687, 2005.

[18] J. Zhang, K. H. Johansson, J. Lygeros, and S. S. Sastry. Zeno hybrid systems. *International Journal of Robust and Nonlinear Control*, 11(5):435–451, 2001.