Optimal Control on Lie Groups

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Talk Outline

1. Review of Optimal Control (Symplectic)
2. Structure of Optimal Controls
3. Optimal Control on SE(2)
4. Optimal Control on SO(3)
5. Optimal Control on SE(3)
6. Concluding Remarks

1. Review of Optimal Control

$$\dot{x} = f(x, u) \quad x \in \Re^n \quad u \in \Re^{n_i}$$

Given x_0,

$$\min \int_0^T L(x, u)dt$$

Define $H(x, u, p) = L(x, u) - p^T f(x, u)$, for $p^T \in \Re^{n*}$.

Optimal Control

$$u^*(x, p) := \arg\min_u H(x, u, p)$$

Optimal Hamiltonian

$$H^*(x, p) = H(x, u^*(x, p), p)$$

Hamilton Jacobi equations

$$\begin{aligned} \dot{x} &= -\frac{\partial H^*}{\partial p}^T \\ \dot{p} &= \frac{\partial H^*}{\partial x}^T \end{aligned}$$

2. Optimal control: Symplectic Version

For $\phi, \psi : T^*\Re^n \to \Re$, define Lie-Poisson bracket $\{\phi, \psi\}$, $T^*\Re^n \to \Re$:

$$\{\psi(x,p), \phi(x,p)\} = \sum_{i=1}^{n} \left(\frac{\partial \psi}{\partial x_i}\frac{\partial \phi}{\partial p_i} - \frac{\partial \phi}{\partial p_i}\frac{\partial \psi}{\partial x_i}\right)$$

Hamilton Jacobi Equation

$$\dot{\psi} = -\{\psi, H^*\}$$

For vector field f, define the Momentum Function as a function $T^*\Re^n \to \Re$

$$P(f) = < p, f(x) >$$

The momentum function is a Lie algebra homomorphism:

$$\{P(f), P(g)\} = -P([f,g])$$

3. Drift Free Control Systems

Consider steering from $x(0) = x_i$ to $x(T) = x_f$ the system

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m$$

Minimize $\frac{1}{2}\int_0^T |u|^2 dt$.

$$\begin{aligned} H(x,u,p) &= \tfrac{1}{2}|u|^2 - \Sigma_{i=1}^m < p, g_i > u_i \\ &= \tfrac{1}{2}|u|^2 - \Sigma_{i=1}^m P_i u_i \end{aligned}$$

and $u_i^*(x,p) = P_i$ and

$$H^*(x,p) = -\frac{1}{2}\sum_{i=1}^m P_i^2$$

Differentiating we get

$$\frac{d}{dt}\begin{bmatrix} u_1^* \\ \vdots \\ u_m^* \end{bmatrix} = \begin{bmatrix} 0 & \{P_1, P_j\} & \{P_1, P_m\} \\ \cdots & \{P_i, P_j\} & \cdots \\ \{P_m, P_1\} & \{P_m, P_j\} & 0 \end{bmatrix} \begin{bmatrix} u_1^* \\ \vdots \\ u_m^* \end{bmatrix}$$

4. Unitary Optimal Controls

From previous

$$\dot{u}^* = \Omega(P)u^*$$

with $\Omega(P) \in so(m)$. Thus,

$$u^*(t) = A(t)u^*(0)$$

with $A(t) \in SO(m)$. Thus, $|u(t)|^2 \equiv |u(0)|^2$.

1. u^* also solves any minimization of the form $\int_0^1 \psi(|u|^2)dt$ for monotone ψ.

2. $\frac{u(t)}{|u(0)|}$ solves the minimum T problem subject to $\Sigma_{i=1}^m |u_i|^2 \leq 1$.

5. More Examples

1. If $m = 2$,

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\omega :=< p, [g_1, g_2] >, \psi(t) = \int_0^t \omega(t)dt$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \cos\psi(t) & -\sin\psi(t) \\ \sin\psi(t) & \cos\psi(t) \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

2. Heisenberg Control Lie Algebra:

$$g_1 = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}\ g_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2p_3(t) \\ -2p_3(t) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Since $p_3(t) \equiv p_3(0)$, optimal inputs are sinusoids.

6. Systems with Drift

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

Minimize $\int_0^T |u(t)|^2 dt$ subject to $x(0) = x_i$ and $x(T) = x_f$.

$P_i :=< p, g_i > \qquad P(f) :=< p, f >$ yields

$$u_i^* = P_i \quad H^*(x,p) = -P(f) - \frac{1}{2} \sum_{i=1}^{m} P_i^2$$

so that

$$\begin{bmatrix} \dot{u}_1^* \\ \vdots \\ \dot{u}_m^* \end{bmatrix} = \begin{bmatrix} \{P_1, P(f)\} \\ \vdots \\ \{P_m, P(f)\} \end{bmatrix} + \Omega(P) \begin{bmatrix} u_1^* \\ \vdots \\ u_m^* \end{bmatrix}$$

Similar formulas for higher order derivatives can be obtained.

7. Optimal Control on $SE2$

Planar Elastica

Pursuit problems (Dubbins), Hilare

Lagrangian $u_1^2 + u_2^2$ is independent of g:

$$\begin{bmatrix} \dot{R} & \dot{x} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -u & 1 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With optimal control $u^* = P_3$, $H^*(g,p) = -P_1 - \frac{1}{2}P_3^2$.

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} = - \begin{bmatrix} \{P_1, H^*\} \\ \{P_2, H^*\} \\ \{P_3, H^*\} \end{bmatrix} = \begin{bmatrix} P_2P_3 \\ -P_1P_3 \\ P_2 \end{bmatrix}$$

Casimir for $H^* = P_1^2 + P_2^2$.

8. Planar Elastica

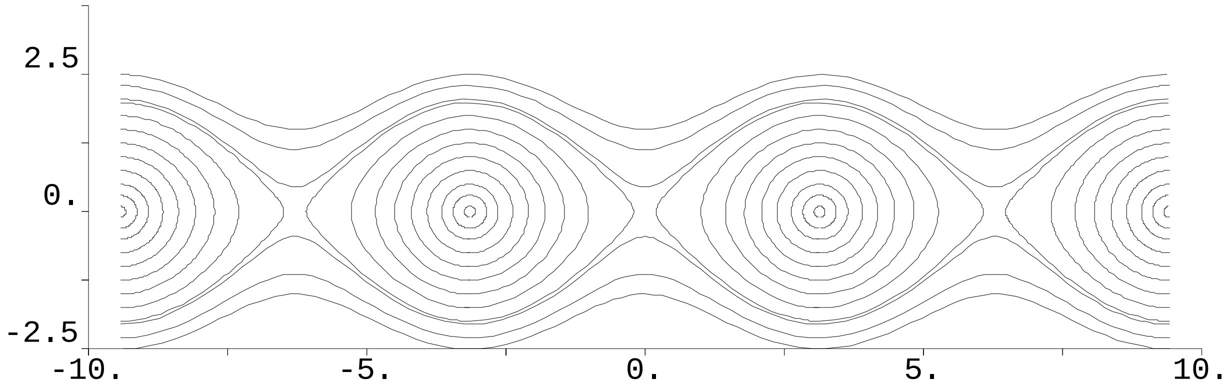
From the Casimir, with $u = P_3$

$$P_1(t) = l\cos\psi(t)$$

$$P_2(t) = l\sin\psi(t)$$

$$P_3(t) = -\dot{\psi}(t)$$

$$\ddot{\psi} - l\sin\psi = 0$$



For $l = 0$, trajectories are circles and straight lines,

for the saddle connection the trajectory is:

9. Optimal Control on $SO(3)$

Steering a satellite with two actuators:

$$\dot{g} = g \begin{bmatrix} 0 & -1 & u_2 \\ 1 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Lagrangian (independent of g) $c_1 u_1^2 + c_2 u_2^2$

Optimal controls are

$$u_1 = \frac{1}{c_1} P_1 \qquad u_2 = \frac{1}{c_2} P_2$$

$$\begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{pmatrix} = \begin{pmatrix} P_2 - \frac{1}{2c_2} P_2 P_3 \\ -P_1 + \frac{1}{2c_1} P_1 P_3 \\ \frac{c_1 - c_2}{2c_1 c_2} P_1 P_2 \end{pmatrix}$$

For $c_1 = c_2$, Casimirs are:

$$k_1 = P_1^2 + P_2^2 \qquad k_2 = P_3$$

10. Optimal Controls on $SO(3)$

Using Casimir's, $k_1 = P_1^2 + P_2^2$, we get

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} \sqrt{k_1}\cos\theta(t) \\ \sqrt{k_1}\sin\theta(t) \\ k_3 \end{pmatrix}$$

with $\dot{\theta}(t) = (1 - k_2/2)$. This yields that optimal inputs are sinusoids

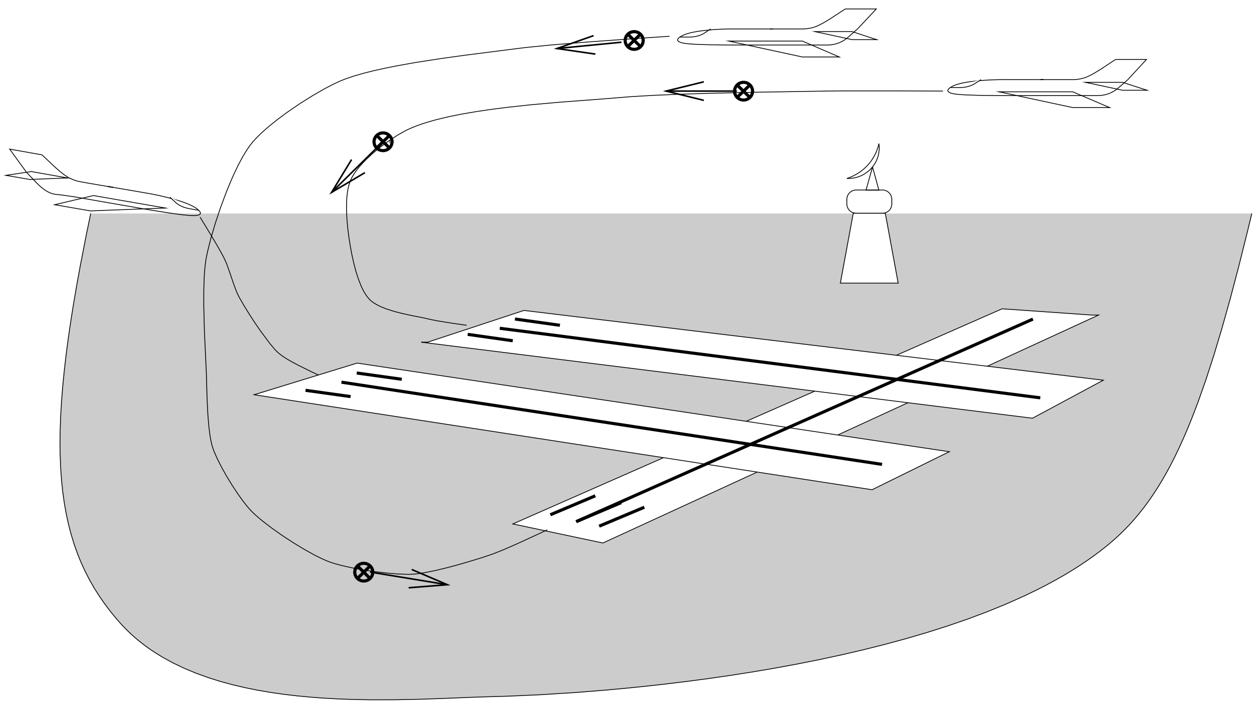
$$u_1[u_2](t) = \sqrt{k_1}\cos[\sin]((1 - k_2/2)t + k_3)$$

In turn with $a = \sqrt{k}_1, b = (1 - k_2/2), c = k_3$,

$$g(t) = \exp[tS(a\ 0\ b)^T)]\ \exp[-(bt + c)e_3]$$

$S(x)$ is the $so(3)$ matrix associated with $x \in \Re^3$.

11. Air Traffic Control Problem



Kinematic Path planner

$$\begin{bmatrix} \dot{R} & \dot{x} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -u_3 & u_2 & 1 \\ u_3 & 0 & -u_1 & 0 \\ -u_2 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $R \in SO(3), x \in \Re^3$.

If $u_2 = 0$ (no roll), Frenet Serret equations.

12. Structure of Optimal Controls

With Lagrangian $u_1^2 + u_2^2 + u_3^2$, we get optimal controls:

$u_1 = P_1, u_2 = P_2, u_3 = P_3$, with the equations

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} = \begin{bmatrix} P_4 \\ P_5 \\ P_6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \dot{P}_4 \\ \dot{P}_5 \\ \dot{P}_6 \end{bmatrix} = \begin{bmatrix} P_4 \\ P_5 \\ P_6 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

EQUATIONS OF THE LAGRANGE TOP!

with Casimirs:

$$k_1 = \sum_{i=4}^{6} P_i^2 \quad k_2 = \sum_{i=1}^{3} P_i P_{i+3} \quad k_3 = P_1$$

13. Helical Solutions

$$P_4(0) = P_5(0) = P_6(0) = 0 \Rightarrow P_4(t) = P_5(t) = P_6(t) = 0$$

$$P_i(t) = 0, i = 4, 5, 6 \Rightarrow P_1(t) = c_1, P_2(t) = c_2, P_3(t) = c_3$$

Thus, the inputs $u_i^* = P_i$ for $i = 1, 2, 3$ are constant

and the optimal trajectories are:

$$\dot{g} = g \begin{bmatrix} 0 & -c_3 & c_2 & 1 \\ c_3 & 0 & -c_1 & 0 \\ -c_2 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These trajectories are helical: including straight lines

and circles as degenerate cases: planning trajectories

of choice for air-traffic controllers.

14. General Optimal Control Solution

Theorem: Using the Casimirs

$$k_1 = P_4^2 + P_5^2 + P_6^2$$

$$k_2 = P_1P_4 + P_2P_5 + P_3P_6$$

$$k_3 = P_1$$

the optimal controls are:

$$u_1 = k_3$$

$$u_2 = r(t)\cos\theta(t)$$

$$u_3 = r(t)\sin\theta(t)$$

$$r(t) = \sqrt{2H^* - k_3^2 - 2P_4(t)}$$

$$\dot{\theta}(t) = \frac{k_1 - k_3P_4(t)}{2H^* - k_3^2 - P_4(t)}$$

$P_4(t)$ can be solved for using Weierstrass P functions.

15. Weierstrass Functions

$$P_4(t) = 2^{\frac{1}{3}}\mathcal{P} + \frac{1}{3}H$$

where $\mathcal{P}$ is a Weierstrass P function:

$$\dot{\mathcal{P}} = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$$

with

$$g_2 = 2^{\frac{4}{3}}(H^2 - k_1^2 + k_2k_3)$$

$$g_3 = (k_2 - 1/3H^2) + (1/9H^2 - k_1^2)(k_3^2 - 4/3H)$$

16. Concluding Remarks

1. Solved Optimal Control for left invariant vector fields on $SO(2), SO(3), SE(3)$ with Lagrangian independent of x, using Hamiltonian reduction.

2. Quantum control needs $SU(n)$.

3. Numerical issues in solving optimal control curves.

4. Lagrangian reduction also possible (Koon-Marsden).