The connection between information theory and networked control

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based in part on joint work with students:
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Wireless Foundations
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Networked Control Systems

- Systems, sensors, and users connected with network links over noisy channels.
- Signals evolve in real time and the communication links carry ongoing and interacting streams of information.
- Holistic approach: overall cost function.
Ho, Kastner, and Wong (1978)

“...sporadic and not too successful attempts have been made to relate Shannon’s information theory with feedback control system design.”
Shannon tells us

Separate source and channel coding

Fig. 1 — Schematic diagram of a general communication system.
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- But delay is the price of reliability.

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What is this relationship since delays hurt control?
Outline

1. A bridge to nowhere? From control to information theory.
   - A simple control problem
   - A connection to information theory
   - Fixing information theory and filling in the gaps.

2. Coming back to the control problem
   - What is wrong with random coding
   - The role of noiseless feedback

3. Taking control thinking to the forefront of information theory.
   - The “holy grail” problem
   - Control thinking to the rescue!
A simple distributed control problem

\[ X_{t+1} = \lambda X_t + U_t + W_t \]

- Unstable \( \lambda > 1 \), bounded initial condition and disturbance \( W \).
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- Goal: Performance = \( \sup_{t>0} E[\|X_t\|_\eta] \leq K \) for some target \( K < \infty \).
Review: Entirely noiseless channel

Window known to contain $X_t$

will grow by factor of $\lambda > 1$

Sending $R$ bits, cut window by a factor of $2^{-R}$

Encode which control $U_t$ to apply

grows by $\frac{\Omega}{2}$ on each side

giving a new window for $X_{t+1}$

As long as $R > \log_2 \lambda$, we can have $\Delta$ stay bounded forever.
The separation-principle oriented program

- Use entropy and mutual information
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  - Rate-distortion theory can be developed
- Get tight upper bounds and architectures?
The rate-distortion part

- "Sequential" Rate-distortion (obeys causality)
- Rate-distortion curve (non-causal)
- Stable counterpart (non-causal)
Consider a system with

- $\lambda = 2$ for the dynamics
- Real packet-drop channel ($C = \infty$)

$$Z_t = \begin{cases} 
Y_t & \text{with Probability } \frac{1}{2} \\
0 & \text{with Probability } \frac{1}{2}
\end{cases}$$
How bad can entropic bounds be?

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Under stochastic disturbances, the variance of the state is asymptotically infinite. (*St. Petersburg Lottery Style*)
Delay-universal \textit{(anytime)} communication

\[
B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \quad B_6 \quad B_7 \quad B_8 \quad B_9 \quad B_{10} \quad B_{11} \quad B_{12} \quad B_{13}
\]

\[
\begin{array}{cccccccccccccccc}
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\]

fixed delay \(d = 7\)
Delay-universal \textit{(anytime)} communication

\begin{itemize}
  \item Fixed-delay reliability $\alpha$ is achievable if there exists a sequence of encoder/decoder pairs with increasing end-to-end delays $d_j$ such that
  \[ \lim_{j \to \infty} \frac{-1}{d_j} \ln P(B_i \neq \hat{B}_i^j) = \alpha. \]
\end{itemize}
Delay-universal (*anytime*) communication

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\[ \text{fixed delay } d = 7 \]

\( \alpha \) is achievable *delay-universally* or in an *anytime fashion* if a single encoder works for all sufficiently large delays \( d \).
Delay-universal (*anytime*) communication

\[ Y_1 \cdots Y_{26} \]

\[ Z_1 \cdots Z_{26} \]

\[ \widehat{B}_1 \cdots \widehat{B}_9 \]

fixed delay \( d = 7 \)

- The anytime capacity \( C_{\text{any}}(\alpha) \) is the supremal rate at which reliability \( \alpha \) is achievable in a delay-universal way.
Separation theorem for scalar control

*Necessity:* If a scalar system with parameter \( \lambda > 1 \) can be stabilized with finite \( \eta \)-moment across a noisy channel, then the channel with noiseless feedback must have

\[
C_{\text{any}}(\eta \ln \lambda) \geq \ln \lambda
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In general: If \( P(|X| > m) < f(m) \), then \( \exists K : P_{\text{error}}(d) < f(K\lambda^d) \)
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Captures stabilization only.
Some easy implications

If we want $P(|X_t| > m) \leq f(m) = 0$ for some finite $m$, we require zero-error reliability across the channel. Also required (for DMCs) if we want the controller to be finite memory.
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- For generic DMCs, anytime reliability with feedback is upper-bounded:

\[
f(K \lambda^d) \geq \zeta^d
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\[
f(m) \geq K'm^{-\log_2 \left( \frac{1}{\zeta} \right) \log_2 \lambda}
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A controlled state can have at best a power-law tail.
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- Almost-sure stabilization for $W_t = 0$ follows by simple time-varying transformation.
Asymptotic communication problem hierarchy

The easiest: Shannon communication

- Asymptotically: a single figure of merit $C$
- Equivalent to most estimation problems of stationary ergodic processes with bounded distortion measures.
- Feedback does not matter.
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- Intermediate families: Anytime communication
  - Multiple figures of merit: $(\vec{R}, \vec{\alpha})$
  - Feedback case equivalent to stabilization problems
  - Related nonstationary estimation problems fall here also
  - Does feedback matter?

- Hardest level: Zero-error communication
  - Single figure of merit $C_0$
  - Feedback matters.
My favorite example: the BEC

- Simple capacity $1 - \delta$ bits per channel use
- With perfect feedback, simple to achieve: retransmit until it gets through
  - Time till success: Geometric($1 - \delta$)
  - Expected time to get through: $\frac{1}{1 - \delta}$
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Classical bounds
- Sphere-packing bound $D(1 - R \parallel \delta)$
- Random coding bound $\max_{\rho \in [0,1]} E_0(\rho) - \rho R$
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What happens with feedback?
BEC with feedback and fixed *blocks*

- At rate $R < 1$, have $Rn$ bits to transmit in $n$ channel uses.
- Typically $(1 - \delta)n$ code bits will be received.
BEC with feedback and fixed blocks

- At rate $R < 1$, have $Rn$ bits to transmit in $n$ channel uses.
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  - $D(1 - R \| \delta)$
- Dobrushin-62 showed that this type of behavior is common: $E^+(R) = E_{sp}(R)$ for symmetric channels.
BEC with feedback and fixed delay

- $R = \frac{1}{2}$ example:

- Birth-death chain: positive recurrent if $\delta < \frac{1}{2}$
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$$P(D \geq d) = P(L > \frac{d}{2}) = K\left(\frac{\delta}{1 - \delta}\right)^d$$
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**Block-coding is misleading!**
Without feedback: $E^+(R)$ continues to be a bound.

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- Use it to construct a block-code with blocksize $n \gg d$
- Genie-aided decoder: has the truth of all bits before $i$
Pinsker’s bounding construction explained

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  - Apply a change of measure argument
Using $E_{sp}$ to bound $\alpha^*$ in general

The block error probability is like $\exp(-\alpha(1 - \lambda)n)$ which cannot exceed the Haroutunian bound $\exp(-E_{sp}(\lambda R)n)$
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![Table and Diagram]

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$$\alpha^*(R) \leq \frac{E_{sp}(\lambda R)}{1 - \lambda}$$
Using $E_{sp}$ to bound $\alpha^*$ in general

<table>
<thead>
<tr>
<th>Past behavior</th>
<th>Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda n$</td>
<td>$(1 - \lambda)n$</td>
</tr>
<tr>
<td>$\frac{\lambda}{1-\lambda}d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$\lambda R'n$</td>
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- The error events involve both the past and the future.
Uncertainty-focusing bound for symmetric DMCs

Minimize over $\lambda$ for symmetric DMCs to sweep out frontier by varying $\rho > 0$:

$$R(\rho) = \frac{E_0(\rho)}{\rho}$$

$$E_a^+(\rho) = E_0(\rho)$$

Using the Gallager function:

$$E_0(\rho) = -\max_q \ln \sum_j \left( \sum_i q_i p_{ij}^{1+\rho} \right)^{1+\rho}$$
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Same form as Viterbi’s “convolutional coding bound” for constraint-lengths, but a lot more fundamental!
Upper bound tight for the BEC with feedback

![Graph showing error exponent vs rate for the BEC with feedback. The x-axis represents rate (in bits), and the y-axis represents error exponent (base 2). The graph includes multiple curves representing different error exponents.]
Implications for scalar moment stabilization
Implications for scalar moment stabilization

![Graph showing the relationship between moments stabilized and open-loop unstable gain.](image_url)
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   - The “holy grail” problem
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- It diverges locally whenever the channel misbehaves.
- Semi-reasonable implementation complexity.

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Controller and Computations

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- Classical results tell us that with appropriate bias, achieve $E_r(R_{branch})$ for error probability and hence power-law in state.
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At the cost of only finite expected computation.
Catch up “all-at-once” phenomenon

Simulation Parameters:
\( \lambda = 1.1 \)
\( \varepsilon = 0.05 \)
\( \Omega = 2.0 \)
\( \Delta = 5000.0 \)
Bias = 0.55
\( T = 10 \)
100,000 Blocks
17 seconds to run

Rate = 0.317
Capacity = 0.71
Although we are doing better than exponential growth, we still have power laws on both sides.

What if we needed a finite speed computer in the controller?
Although we are doing better than exponential growth, we still have power laws on both sides.

What if we needed a finite speed computer in the controller?

Bad news:
- Assume 0 control applied if we can not decode yet.
Although we are doing better than exponential growth, we still have power laws on both sides.

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Bad news:
- Assume 0 control applied if we can not decode yet.
- Power law for comp. implies power low for waiting.
- Exponentially rare doubly exponentially bad states!
How to hit the higher bound?
How to hit the higher bound?
Fortified channels

Noisy forward channel uses

Fortification" noiseless forward channel uses

Some mix of noisy and noiseless channels
Fortified channels

Noisy forward channel uses

"Fortification" noiseless forward channel uses

- Some mix of noisy and noiseless channels
- Is it all or nothing?
Noiseless channel can enable event-based sampling

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Need to allow for gradual progress during bad periods.
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- Need to allow for gradual progress during bad periods.
- Use the noiseless channel for supervisory information:
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Use the noiseless channel for supervisory information:
  ▶ Have the observer do event-based “sampling” of the state.
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Use the noiseless channel for supervisory information:

- Have the observer do event-based "sampling" of the state.
- "Quantization net" grows as needed, but has only $e^{nR}$ boxes.

Outer net to quantize and encode the state

Inner catchment area to resample the state
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Use the noisy channel for variable-length block-coding.

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Why gradual progress is better: intuition
Why this works: proof strategy

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  - Erasure probability $\exp(-E_0(\rho))$
Outline

1. A bridge to nowhere?
   - A simple control problem
   - A connection to information theory
   - Fixing information theory and filling in the gaps.

2. Coming back to control
   - What is wrong with random coding
   - The role of noiseless feedback

3. Taking control thinking to the forefront of information theory.
   - The “holy grail” problem
   - Control thinking to the rescue!
The “holy grail:” understanding complexity

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![Graph showing the relationship between log₂(P₀) and power, with curves for Uncoded transmission BSC, Shannon Waterfall BSC, and Shannon Waterfall AWGN.]
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  - New short-range applications: swarm behavior, in-home networks, dense meshes, personal-area networks, UWB, between-chip communication, etc.
Decoding power vs communication range

Distance

\( \gamma (\text{dB}) \)

1mm 10mm 1m 100m 10km
Dense linear codes with brute-force decoding

Decoding Power $nR2^{nR}$, Error Prob $2^{-E_{sp}(R,P)n}$
Convolutional codes with Viterbi decoding

Decoding Power $L_c R 2^{L_c R}$, Error Prob $2^{-E_{\text{conv}}(R,P) L_c}$
Convolutional with “magical” sequential decoding

Decoding Power $L_c R$, Error Prob $2^{-E_{\text{conv}}(R,P)L_c}$
Dense linear codes with “magical” syndrome decoding

Decoding Power \((1 - R)nR\), Error Prob \(2^{-E_{sp}(R,P)n}\)
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How to lower-bound the number of iterations?

- Key concept: decoding
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- Need to lower-bound average probability of bit error in terms of \( n \).
- **Key insight:** \( n \) is playing a role analogous to delay.
A local “sphere-packing” bound for the AWGN

Decoding neighborhood size $n \leq 1 + (\alpha + 1)\alpha^{i-1} \approx \alpha^i$.

$$\langle P_e \rangle \geq \sup_{\sigma^2_G > \sigma^2_P} \mu(n): C(G) < R$$

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Double-exponential potential return on investments in decoding power!
Waterslide curves for general AWGN case

\[
\log_{10}(\langle P_e \rangle) = \begin{cases} 
\gamma = 0.4 \\
\gamma = 0.3 \\
\gamma = 0.2 \\
\text{Shannon limit}
\end{cases}
\]