Trade-off of lossless source coding error exponents

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Motivation

- Sensor networks: control, communication and information theory
  - Distributed control
  - Unstable process with noisy feedback loop
Stabilizing an unstable system with noisy feedback

- Scalar case \( X_{t+1} = \lambda X_t + U_t + W_t \) (Sahai/Mitter 2006)
  - \( E(|X_t|^{\eta}) < \infty \), for all \( t \) iff delay exponent \( E(\log_2(\lambda)) > \eta \log_2(\lambda) \)
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  - $E(|X_t|^\eta) < \infty$, for all $t$ iff delay exponent $E(\log_2(\lambda)) > \eta \log_2(\lambda)$

- Vector case $\vec{X}_{t+1} = A\vec{X}_t + \vec{U}_t + \vec{W}_t$: $A$ eigenvalues $\lambda_1, \lambda_2, \ldots$
  - Different error exponents for different streams
Related work

- Messages requiring different reliability
- Sub-messages requiring different reliability
- This talk is about multi-stream trade-offs.
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  - Csiszar ’80
- Sub-messages requiring different reliability
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  - Kudryashov ’79 and Sahai/Draper ’06, ’08
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- Sub-messages requiring different reliability
  - Priority-encoded transmission (Luby et al ’96, Boucheron/Salamatian ’00)
  - Multiterminal coding error-exponent trade-offs (Weng et al ’08, Kaspi et al ’08, Etkin et al ’08)

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Outline

1 Motivation and related work

2 Main Results:
   - Trade-off for block source coding: a complete characterization
   - Trade-off for streaming source coding: inner and outer bounds
   - Trade-off for BEC with feedback and its control implications

3 Conclusions
Fixed-length block coding for i.i.d. source $x \sim P$

$$x_1^n \in \mathcal{X}^n \rightarrow b_1^{nR} \in \{0, 1\}^{nR} \rightarrow \hat{x}_1^n \in \mathcal{X}^n$$
Block source-coding error exponents

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- Performance criteria: $\Pr(\hat{x}_1^n \neq x_1^n)$ small for large $n$ (lossless)
Block source-coding error exponents

- Fixed-length block coding for i.i.d. source \( x \sim P \)

\[
x^n_1 \in \mathcal{X}^n \rightarrow b^n_{1} \in \{0, 1\}^{nR} \rightarrow \hat{x}^n_1 \in \mathcal{X}^n
\]

- Performance criteria: \( \Pr(\hat{x}^n_1 \neq x^n_1) \) small for large \( n \) (lossless)

- Entropy and error exponent:
  - \( R > H(P) \)
  - \( \Pr(\hat{x}^n_1 \neq x^n_1) \sim 2^{-nE_{\text{block}}(R)} \) for optimal coding

\[
E_{\text{block}}(R) = \min_{Q : H(Q) \geq R} D(Q \| P)
\]
Two sources $X$ and $Y$ share $R$ bits/sec
Error-exponent region (block coding)

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- Error-exponent region
  \[ \{ (E_x, E_y) : \Pr(\hat{x}_1^n \neq x_1^n) \sim 2^{-nE_x} \text{ and } \Pr(\hat{y}_1^n \neq y_1^n) \sim 2^{-nE_y} \} \] for some coding scheme
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Naive outer bound (converse):
Error-exponent region (block coding)

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- Naive outer bound (converse):
  - $E_x \leq E_{block(x)}(R)$, all for one source
  - Either $E_x \leq E_{block(xy)}(R)$ or $E_y \leq E_{block(xy)}(R)$
A non-convex region.
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Suffices to show $B$ is achievable.
A non-convex region.

Suffices to show $B$ is achievable
  - Encoder knows source before encoding.
A non-convex region.

Suffices to show $B$ is achievable

- Encoder knows source before encoding.
- Transmit high priority source if cannot send both
Beyond block coding

- Block coding: $x_1^n$ known at time $-1$, block error probability
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- Real-time coding with delay constraints (Anytime)
  - Causality: source symbols $x_1, x_2, \ldots, x_k, \ldots$ streaming into the encoder
  - End-to-end system delay $\Delta$
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Symbol-wise decoding error: $\Pr(x_i \neq \hat{x}_i(i + \Delta)) \sim 2^{-\Delta E(R)}$

Focusing bound $E(R) = \inf_{\alpha > 0} \frac{1}{\alpha} E_{block}((\alpha + 1)R)$
Error-exponent region (streaming)

- Two streaming sources $X$ and $Y$ share $R$ bits/sec
- Error exponent region

$$\{(E_x, E_y) : \Pr(x_t \neq \hat{x}_t(t+\Delta)) \sim 2^{-\Delta E_x} \text{ and } \Pr(y_t \neq \hat{y}_t(t+\Delta)) \sim 2^{-\Delta E_y}\}$$
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- Inner and outer bounds
Proof ideas

- Generalization of p-to-p source coding with delay (Chang 2006)
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Outer bound: uncertainty-focusing bound for two sources

\[
x \quad (1 + \beta)n
\]

\[
y \quad \alpha n \quad n
\]
Two bitstreams $R_1$ and $R_2$ share a BEC($\beta$)
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Generalization of point to point case (Sahai ’08)
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Inner bounds (better than Sahai ’00) and outer bounds
Implications for control systems

- $\mathbf{X}_{t+1} = A \mathbf{X}_t + \mathbf{U}_t + \mathbf{W}_t$: $A$ with eigenvalues $\lambda_1, \lambda_2$
- Control-feedback channel: BEC with feedback
- Inner and outer bounds on stabilizable($\lambda_1, \lambda_2$) for $\beta = 0.1, \eta = 2$

Conclusions

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  - Inner and outer bounds for anytime BEC with feedback
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- Future directions
  - Tightening the inner and outer bounds
  - Extending to general channels
  - Trade-off of lossy source-coding error exponents (block and streaming)