The “Hallucination Bound” for the BSC

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Outline

1. Motivation and review
2. Block-coding converse: minimum error probability
3. Streaming converse: bit error probability
Feedback is pointless at high rates. (Dobrushin and Haroutunian)

Hard decision regions cover space
Refuse to decide when ambiguous

Decision regions catch the typical sets only
Review: Fixed blocks: Forney-68

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- Can interpret as expected block-length

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- Decision regions catch the typical sets only
- 1 bit feedback can request retransmissions
- Can interpret as expected block-length
- No converse except at zero rate

Refuse to decide when ambiguous
Showed $C_1(1 - \frac{R}{C})$ was a bound where $C_1 = \max_{i,j} D(p_i \| p_j)$

Considered expected stopping time and used Martingale arguments.
Review: Fixed blocks, Soft deadlines: Burnashev-76

Block length $n$

Data Transmission $\lambda n$

Ack/Nak $(1 - \lambda)n$

Enc Decision feedback $= \hat{m}$ Dec

possible retransmissions
Streaming: an opportunity presents itself

What if we only sent NAKs when needed?
Sliding blocks with collective punishment only (Kudryashov-79)

- Make packet length $n$ much smaller than soft deadline $\Delta$.
- A NAK collectively denies the past $\frac{\Delta}{n} - 1$ packets.
- Error only if $\frac{\Delta}{n} - 1$ NAKs are all missed.
Reason for hope: Csiszar’s result ’80

Can pack-in control messages at lower rates and give each subset their own random-coding bound!
Even simpler: need only one special message
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Specialize to BSC case

Use all zero for NAK

\[ q_y = q(1-p) + (1-q)p \]

Gap

codeword composition \[ \rightarrow \] output composition

BSC \((p)\)
Specialize to BSC case

- Use all zero for NAK
- Use composition $q$ code for data:
  $R < H(q_y) - H(p)$

![Diagram showing the relationship between codeword composition, BSC(p), and output composition with a gap between $q$ and $p$.]
Specialize to BSC case

- Use all zero for NAK
- Use composition $q$ code for data:
  $$ R < H(q_y) - H(p) $$
- Probability of missed NAK is
  $$ 2^{-nD(q_y||p)} $$
Resulting exponents

- Delay exponent
- Burnashev exponent
- Forney exponent
- Sphere packing exponent
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- What about for the minimum probability of error with feedback?
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- Exponent at most: \(\log_2 \frac{1}{p} - R\)
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- Each normal message needs $2^{nH(p)}$ to decode to it.
- Must claim $2^{n(R+H(p))}$ volume.
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- Matches achievability!

\[ B(p) \]

\[ + \]

\[ \begin{array}{c}
\text{codeword composition} \\
\text{BSC (p)} \\
\text{output composition}
\end{array} \]

\[ q \]

\[ q_y \]

\[ 0 \]

\[ 0.5 \]

\[ 1 \]

\[ p \]

\[ \text{Gap} \]

\[ q (1-p) + (1-q) p \]
Generalizes to general symmetric channels

- Consider fixed block-length and moderate probability of correct decoding for “many” regular codewords.
- Berger’s source-coding game reveals that typical channel outputs can only reach $\mathcal{P}_y$ output types.
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$$\approx 2^{-n(D(p_y||\tilde{p}_y) + H(p_y))} 2^{nR + nH(Y|X)}$$
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- Allow convex hull of
  \[
  E_{hal}(R) = \max_{I(p_x,P) \geq R} \max_{\tilde{p}_y \in \mathcal{P}_y} [D(P(p_x)||\tilde{p}_y) + (I(p_x, P) - R)]
  \]
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- On Friday Borade, et al will show a more general proof that holds with expected block-length.
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The basic intuition

- Bit Enters $\Delta$
- Bit is decoded

- If we hallucinate for $\Delta$, we will miss our opportunity to decode.
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- Challenge: overlaps with other bit-footprints.
The idea of the proof

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Count the volume of normal decoding regions in two different ways.
  - As a big block code based on correct decoding.
  - With a tree-structure based on the desired exponent.

Technical condition: impose sequentiality on the code: we can usually guess the answer even before the deadline runs out.
Conclusions

- The “Hallucination Bound” is the probability that the decoder imagines that everything is normal despite your best efforts to tell it otherwise.

- This corresponds to the best probability of error for a special message in the fixed block-code setting.
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Future work

- Eliminate the technical condition for the streaming case.
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Future work
- Eliminate the technical condition for the streaming case.
- Get a two-way “Hallucination bound” for the case of noisy feedback.