The necessity and sufficiency of anytime capacity for control over a noisy communication link

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Outline

1. Introduction and problem setup
2. Why Shannon capacity and mutual information are not enough
3. Necessity of anytime reliability
4. Consequences: power-laws vs zero-error requirements
5. Sufficiency of anytime reliability
6. Conclusions and extensions
Shannon’s Formulation of Communication

Large end-to-end delay is permitted.

“Meaning” introduced through end-to-end distortion measure.

“... can be pursued further and is related to a duality between past and future and the notions of control and knowledge. Thus we may have knowledge of the past and cannot control it; we may control the future but have no knowledge of it.” — Claude Shannon 1959

Many people have taken up the challenge. Prior talk gave some pointers. See Sep. 2004 *Trans. on Auto. Control* for more.
Our simple scalar distributed control problem

\[ X_{t+1} = \lambda X_t + U_t + W_t \]

- Unstable \( \lambda > 1 \), bounded initial condition and disturbance \( W \).
- Goal: Stability = \( \sup_{t>0} E[|X_t|^\eta] \leq K \) for some \( K < \infty \).
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Is Shannon capacity all we need? (Review)

- Consider a system with
  - $\lambda = 2$ for the dynamics
  - noisy channel that sometimes drops packets but is otherwise noiseless (Real erasure channel)

$$Z_t = \begin{cases} 
Y_t & \text{with Probability } \frac{1}{2} \\
0 & \text{with Probability } \frac{1}{2}
\end{cases}$$

- No other constraints, so design is obvious: $Y_t = X_t$ and $U_t = -\lambda Z_t$

- Resulting closed loop dynamics:

$$X_{t+1} = \begin{cases} 
W_t & \text{with Probability } \frac{1}{2} \\
2X_t + W_t & \text{with Probability } \frac{1}{2}
\end{cases}$$
Is the closed-loop system stable?

\[ X_{t+1} = \begin{cases} W_t & \text{with Probability } \frac{1}{2} \\ 2X_t + W_t & \text{with Probability } \frac{1}{2} \end{cases} \]

- i.i.d. erasures mean arbitrarily long stretches of erasures are possible, though unlikely.
  - System is not guaranteed to stay inside any box.
  - Under stochastic disturbances, the variance of the state is asymptotically infinite.

- For worst case disturbances \( W_t = 1 \), the tail probability is dying off as \( P(|X| > x) \approx \frac{K}{x} \).

- Meanwhile, \( C = \infty \)!
Run same plant $\bar{X}$ over noiseless channel

- Window known to contain $\bar{X}_t$
- Will grow by factor of $\lambda > 1$
- Sending $R$ bits, cut window by a factor of $2^{-R}$

Encode which virtual control $\bar{U}_t$ to apply

grows by $\frac{\Omega}{2}$ on each side

giving a new window for $\bar{X}_{t+1}$

As long as $R > \log_2 \lambda$, we can have $\Delta$ stay finite forever.
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What is needed: key intuition

- Break state $X$ into sum of $\tilde{X}$ (response to disturbance) and $\hat{X}$ (response to control)
- Suppose $\lambda = 2$ and so $\tilde{X}_t = \sum_{i=0}^{t} 2^i W_{t-1}$
- Assume $W_j$ either 0 or 1
- In binary notation: $\tilde{X}_t = W_0 W_1 W_2 \ldots W_{t-1}.00000\ldots$
- If $-\tilde{X}_t$ is close to $X_t$, their binary representations likely agree in all the high-order bits.
  - High-order bits represent earlier disturbances.
  - Typically, to get a difference at the $W_{t-d}$ level, we have to be off by about $2^d$.

Stabilization implies communicating bits reliably.
Anytime reliable transmission

- Have a fixed encoder, but let the decoder be parametrized by the delay. Want a good estimate “anytime” we ask for one.
“Reliable Transmission” means every bit is eventually correctly received. Parametrize by the rate at which the probability of bit error \( P(S_{t-d} \neq \hat{S}_{t-d}(t)) \) goes to zero as delay \( d \) increases.

\[
C_{\text{anytime}}(\alpha) = \sup \left\{ R \mid \begin{array}{l}
\exists (\mathcal{E}, \mathcal{D}, K) \forall d > 0 \\
\text{Rate} = R, \text{Delay} = d, \\
P_{\text{error}}(\mathcal{E}, \mathcal{D}, d) \leq K2^{-\alpha d}
\end{array} \right\}
\]

Related to information-theoretic error exponents, but represent a “delay universal” variation.
BEC feedback anytime reliability vs $E_{sp}$

\[
C_{\text{anytime}}(\alpha) = \frac{\alpha}{\alpha + \log_2\left(\frac{1-\epsilon}{1-\epsilon 2^\alpha}\right)}
\]
Separation theorem for control

Necessity: If a scalar system with parameter $\lambda > 1$ can be stabilized with finite $\eta$-moment across a noisy channel, then the channel with noiseless feedback must have

$$C_{\text{anytime}}(\eta \log_2 \lambda) \geq \log_2 \lambda$$

Sufficiency: If there is an $\alpha > \eta \log_2 \lambda$ for which the channel with noiseless feedback has

$$C_{\text{anytime}}(\alpha) > \log_2 \lambda$$

then the scalar Markov system with parameter $\lambda \geq 1$ with a bounded disturbance can be stabilized across the noisy channel with finite $\eta$-moment by using observers that have noise-free access to the control signals and channel outputs.
Proof Idea: Necessity (details in paper)

- Follow the key intuition and embed data to be communicated into a bounded disturbance.
- Technical tool: Instead of binary, use a Cantor set based mapping.
- Recover data from $-\tilde{X}_t$.
- If $P(|X| > m) < f(m)$, then $\exists K$:

$$P_{\text{error}}(d) < f(K\lambda^d)$$
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What does all this imply?

- If we want \( P(\left| X_t \right| > m) \leq f(m) = 0 \) for some finite \( m \), we require zero-error reliability across the channel.

- For generic DMCs, the anytime reliability with feedback can be upper-bounded and it is at most exponential in nature.

\[
\begin{align*}
    f(K\lambda^d) & \geq \rho^d \\
    f(m) & \geq \rho \frac{\log_2 \left( \frac{m}{K} \right)}{\log_2 \lambda} \\
    f(m) & \geq K' m^{-\frac{\log_2 \frac{1}{\rho}}{\log_2 \lambda}}
\end{align*}
\]

A controlled state can have at best a power-law tail!

- If we just want \( \lim_{m \to \infty} f(m) = 0 \), then just Shannon capacity is required for DMCs.
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Sufficiency construction: nested information case

- Observer pretends it is controlling $\bar{X}$ with the same disturbance but a noiseless channel.
- Anytime code used to communicate $\bar{U}_t$
Controller action: try to track $\bar{X}_t$

$Z_t \xrightarrow{\text{Anytime channel decoder}} \hat{U}_t^*(t) \xrightarrow{\text{Estimate } \hat{X} \text{ of virtual controls impact}} \hat{X}_{t+1}(t)$

$\hat{X}_t \xrightarrow{\text{Internal model for } X} \hat{X}_t \xrightarrow{\text{Multiply by } -\lambda} U_t$

Uncorrected errors $d$ time steps ago cause at most $\lambda^d$ of an impact in controlled state.

$$P(|X_t| > m) \leq K'''m^{-\frac{\alpha}{\log_2 \lambda}}$$
What about imperfect information patterns?

- Do we now need a higher quality channel?
- The only path from the controller to the observer is through the plant.
Make the plant “dance” in a stable way!
How is the output information conveyed

\[ X_{t+1} - \lambda X_t = U_t + W_t \]

Correct answer: \( Z_t = 1 \)

width of bin

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\(|\mathcal{B}| = 4\)

four possible output letters

Adjusted by \( \lambda F(z_{t-1}) \)

Decoding lattice based on past channel outputs

Letter encoded by \( F(z_t) \)

Actual control \( U_t \)

Applied by controller

Adjusted by \( \lambda F(z_{t-1}) \)

at controller

Quantized desired control \( Q(U) \)

must lie on lattice points spaced by \( |\mathcal{B}|\Omega \)
Conclusions and extensions

- Anytime reliability is the right concept for stabilization.
- Extensions:
  - Boundedly noisy observations
  - Memoryless observers for high quality DMCs
  - Continuous time (use nats)
  - Vector-case
    * Each unstable eigenvalue might need a different reliability: channel code must prioritize
    * Intrinsic delays within the plant impose stricter requirements for non-nested information patterns.
Some backup slides
Open Problems

- Need more tools and results on bounding achievable ($\bar{\alpha}, \bar{R}$) regions.
- Bit-pipes fit together into networks naturally, need analogous understanding for networked reliable bit-pipes.
- Anytime (delay-universal) versions of network information theory: Slepian-Wolf, MAC, Broadcast, Relay, etc.
- Better understand the performance-loss caused by the communication constraint — starting with the gap between $D_{seq}(R)$ and $D(R)$.  

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Connecting with control and estimation

- Classical LQG theory: Everything linear and optimal in quadratic sense. No need for information theory, but nothing generalizes.

- Witsenhausen’s 1968 “counterexample:” Nonclassical information patterns cause trouble even in LQG — need to both signal and control simultaneously.

  “[The weak results we have are] in sharp contrast with the elaborate results of information theory. The latter deals with an essentially simpler problem, because the transmission of the information is considered independently of its use. . . .

  Efforts to establish a new theory of information, taking optimal cost into account, have not as yet been convincing.” — Hans Witsenhausen 1971
• Use the input bits to drive a source simulator whose output looks like the unstable Markov source \((X_{t+1} = aX_t + W_t)\)

• Controlled system state is like the estimation error since the decoder has access to the controls.
Binary strings and Cantor sets

- Map the input bitstream bijectively into a Cantor set

\[ \hat{X} = \sum_{i=0}^{\infty} S_i (2 + \epsilon_1)^{-i} \]

- Embed a suitably scaled, but growing, Cantor set in the unstable \{X\} process
  - Every value for \(X_t\) corresponds to a specific neighborhood of the Cantor set
  - Use comparisons to recover the original bits from \(\hat{X}_t\). The gaps in the Cantor set give us the ability to distinguish reliably!

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What sense of reliability is achieved?

- The gaps in the Cantor set assure us that all $\hat{X}_t$ that differ in their estimates of $S_{t-d}$ are at a distance of at least $\gamma a^d$ from $X_t$.
  - $|X_t - \hat{X}_t| < \gamma a^d$ implies all bits recovered from $\hat{X}_t$ are correct up through $d$ time steps ago.
  - So $P(S_{t-d} \neq \hat{S}_{t-d}(t)) \leq f(\gamma a^d)$

- Fresh estimates of all bits sent so far.
  - If $E[|\hat{X} - X|^{\eta}]$ is finite, the probability of error on a bit $d$ time-steps ago is at most $K'a^{-\eta d} = K'2^{-(\eta \log_2 a)d}$.
  - The reliability of every bit gets better the more we are willing to wait!