1. Linear neural networks

The purpose of this exercise is to reinforce your understanding of neural networks as mathematical functions that can be analyzed at a level of abstraction above their implementation as a network of computing elements. It also introduces a somewhat surprising property of multilayer linear networks with small hidden layers.

For simplicity, we will assume that the activation function is the same linear function at each node: \( g(x) = cx + d \). (The argument is the same (only messier) if we allow different \( c_i \) and \( d_i \) for each node.)

(a) The outputs of the hidden layer are
\[
H_j = g\left(\sum_k W_{k,j} I_k\right) = c \sum_k W_{k,j} I_k + d
\]
The final outputs are
\[
O_i = g\left(\sum_j W_{j,i} H_j\right) = c \left(\sum_j W_{j,i} \left(c \sum_k W_{k,j} I_k + d\right)\right) + d
\]
Now we just have to see that this is linear in the inputs:
\[
O_i = c^2 \sum_k I_k \sum_j W_{k,j} W_{j,i} + d \left(1 + c \sum_j W_{j,i}\right)
\]
Thus we can compute the same function as the two-layer network using just a one-layer perceptron that has weights \( W_{k,i} = \sum_j W_{k,j} W_{j,i} \) and an activation function \( g(x) = c^2 x + d \left(1 + c \sum_j W_{j,i}\right) \).

(b) The above reduction can be used straightforwardly to reduce an \( n \)-layer network to an \( (n-1) \)-layer network. By induction, the \( n \)-layer network can be reduced to a single-layer network. Thus, linear activation functions restrict neural networks to represent only linear functions.

(c) The original network with \( n \) input and output nodes and \( h \) hidden nodes has \( 2hn \) weights, whereas the “reduced” network has \( n^2 \) weights. When \( h \ll n \), the original network has far fewer weights and thus represents the i/o mapping more concisely. Such networks are known to learn much faster than the reduced network; so the idea of using linear activation functions is not without merit.

2. ML estimation of exponential model

(a) \[
L(X; b) = \prod_{i=1}^{N} \frac{1}{b} e^{-\frac{x_i}{b}} = b^{-N} e^{-\frac{1}{b} \sum_{i=1}^{N} x_i}
\]

(b) \[
l(X; b) = \log(L(X; b)) = -N \log(b) - \frac{1}{b} \sum_{i=1}^{N} x_i
\]
\[
\frac{\partial l}{\partial b}(X; b) = - \frac{N}{b} + \frac{1}{b^2} \sum_{i=1}^{N} x_i
\]
\[
\frac{\partial l}{\partial b}(X; b) = 0 \Rightarrow b = \frac{1}{N} \sum_{i=1}^{N} x_i = \bar{x}
\]