Complex decisions

Chapter 17, Sections 1–3

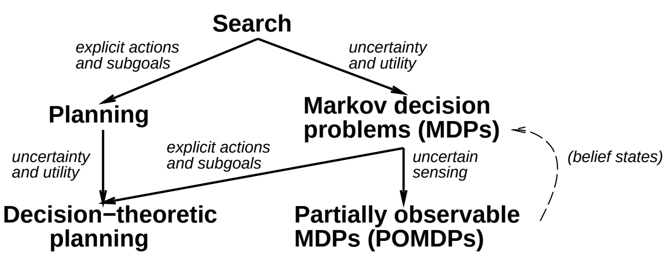
Outline

◇ Sequential decision problems

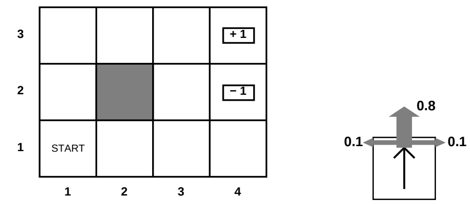
◇ Value iteration

◇ Policy iteration

Sequential decision problems



Example MDP



States $s \in S$, actions $a \in A$

Model $T(s, a, s') \equiv P(s'|s, a) =$ probability that a in s leads to s'

Reward function $R(s)$ (or $R(s, a)$, $R(s, a, s')$)

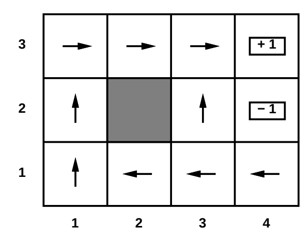
$$= \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

Solving MDPs

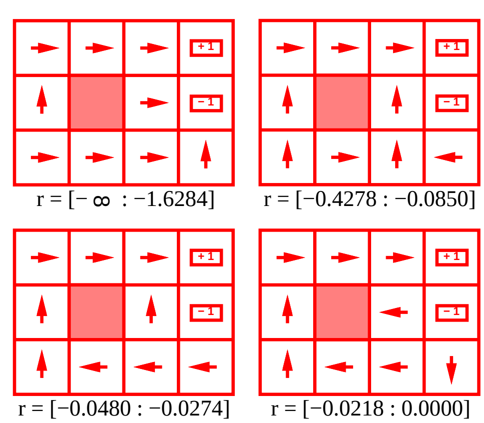
In search problems, aim is to find an optimal *sequence*

In MDPs, aim is to find an optimal *policy* $\pi(s)$
i.e., best action for every possible state s
(because can't predict where one will end up)
The optimal policy maximizes (say) the *expected sum of rewards*

Optimal policy when state penalty $R(s)$ is –0.04:



Risk and reward



Utility of state sequences

Need to understand preferences between *sequences* of states

Typically consider stationary preferences on reward sequences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots] \;\Leftrightarrow\; [r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

Theorem: there are only two ways to combine rewards over time.

1) *Additive* utility function:
$U([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \cdots$

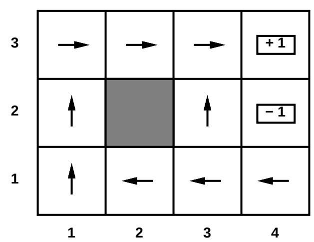
2) *Discounted* utility function:
$U([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots$
where γ is the discount factor

Utility of states

Utility of a *state* (a.k.a. its *value*) is defined to be
$U(s) =$ expected (discounted) sum of rewards (until termination) assuming optimal actions

Given the utilities of the states, choosing the best action is just MEU:
maximize the expected utility of the immediate successors

3	0.812	0.868	0.912	+1
2	0.762		0.660	−1
1	0.705	0.655	0.611	0.388
	1	2	3	4



Utilities contd.

Problem: infinite lifetimes $\Rightarrow$ additive utilities are infinite

1) Finite horizon: termination at a *fixed time* T
$\Rightarrow$ nonstationary policy: $\pi(s)$ depends on time left

2) Absorbing state(s): w/ prob. 1, agent eventually "dies" for any π
$\Rightarrow$ expected utility of every state is finite

3) Discounting: assuming $\gamma < 1$, $R(s) \leq R_{\max}$,

$$U([s_0, \ldots s_\infty]) = \Sigma_{t=0}^{\infty} \gamma^t R(s_t) \leq R_{\max}/(1-\gamma)$$

Smaller $\gamma \Rightarrow$ shorter horizon

4) Maximize system gain = average reward per time step
Theorem: optimal policy has constant gain after initial transient
E.g., taxi driver's daily scheme cruising for passengers

Dynamic programming: the Bellman equation

Definition of utility of states leads to a simple relationship among utilities of neighboring states:

expected sum of rewards
= current reward
$+ \gamma \times$ expected sum of rewards after taking best action

Bellman equation (1957):

$$U(s) = R(s) + \gamma \max_a \Sigma_{s'} U(s') T(s, a, s')$$

$$\begin{aligned} U(1,1) = -0.04 \\ + \gamma \max\{ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), && \textit{up} \\ & 0.9U(1,1) + 0.1U(1,2) && \textit{left} \\ & 0.9U(1,1) + 0.1U(2,1) && \textit{down} \\ & 0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1)\} && \textit{right} \end{aligned}$$

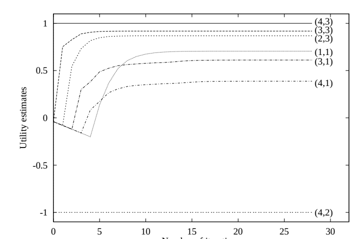
One equation per state = n nonlinear equations in n unknowns

Value iteration algorithm

Idea: Start with arbitrary utility values
Update to make them locally consistent with Bellman eqn.
Everywhere locally consistent $\Rightarrow$ global optimality

Repeat for every s simultaneously until "no change"

$$U(s) \leftarrow R(s) + \gamma \max_a \Sigma_{s'} U(s') T(s, a, s') \qquad \text{for all } s$$



Convergence

Define the max-norm $||U|| = \max_s |U(s)|$,
so $||U - V||$ = maximum difference between U and V

Let U^t and U^{t+1} be successive approximations to the true utility U

Theorem: For any two approximations U^t and V^t

$$||U^{t+1} - V^{t+1}|| \leq \gamma \, ||U^t - V^t||$$

I.e., any distinct approximations must get closer to each other
so, in particular, any approximation must get closer to the true U
and value iteration converges to a unique, stable, optimal solution

Theorem: if $||U^{t+1} - U^t|| < \epsilon$, then $||U^{t+1} - U|| < 2\epsilon\gamma/(1-\gamma)$
I.e., once the change in U^t becomes small, we are almost done.

MEU policy using U^t may be optimal long before convergence of values

Policy iteration

Howard, 1960: search for optimal policy and utility values simultaneously

Algorithm:
- $\pi \leftarrow$ an arbitrary initial policy
- repeat until no change in π
  - compute utilities given π
  - update π as if utilities were correct (i.e., local MEU)

To compute utilities given a fixed π (value determination):

$$U(s) = R(s) + \gamma \sum_{s'} U(s')T(s, \pi(s), s') \qquad \text{for all } s$$

i.e., n simultaneous linear equations in n unknowns, solve in $O(n^3)$

Modified policy iteration

Policy iteration often converges in few iterations, but each is expensive

Idea: use a few steps of value iteration (but with π fixed)
starting from the value function produced the last time
to produce an approximate value determination step.

Often converges much faster than pure VI or PI

Leads to much more general algorithms where Bellman value updates and Howard policy updates can be performed locally in any order

Reinforcement learning algorithms operate by performing such updates based on the observed transitions made in an initially unknown environment

Partial observability

POMDP has an observation model $O(s, e)$ defining the probability that the agent obtains evidence e when in state s

Agent does not know which state it is in
$\Rightarrow$ makes no sense to talk about policy $\pi(s)$!!

Theorem (Astrom, 1965): the optimal policy in a POMDP is a function
$\pi(b)$ where b is the belief state (probability distribution over states)

Can convert a POMDP into an MDP in belief-state space, where
$T(b, a, b')$ is the probability that the new belief state is b'
given that the current belief state is b and the agent does a.
I.e., essentially a filtering update step

Partial observability contd.

Solutions automatically include information-gathering behavior

If there are n states, b is an n-dimensional real-valued vector
$\Rightarrow$ solving POMDPs is very (actually, PSPACE-) hard!

The real world is a POMDP (with initially unknown T and O)