Outline

◇ Rational preferences
◇ Utilities
◇ Money
◇ Multiattribute utilities
◇ Decision networks
◇ Value of information
An agent chooses among prizes \((A, B, \text{etc.})\) and lotteries, i.e., situations with uncertain prizes.

Lottery \(L = [p, A; (1 - p), B]\)

Notation:
- \(A \succ B\) \(A\) preferred to \(B\)
- \(A \sim B\) indifference between \(A\) and \(B\)
- \(A \preceq B\) \(B\) not preferred to \(A\)
Rational preferences

Idea: preferences of a rational agent must obey constraints.
Rational preferences  ⇒
    behavior describable as maximization of expected utility

Constraints:
  Orderability
    \[(A \succ B) \lor (B \succ A) \lor (A \sim B)\]
  Transitivity
    \[(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\]
  Continuity
    \[A \succ B \succ C \Rightarrow \exists p \ (p; A; 1 - p, C) \sim B\]
  Substitutability
    \[A \sim B \Rightarrow [p, A; 1 - p, C] \sim [p, B; 1 - p, C]\]
  Monotonicity
    \[A \succ B \Rightarrow (p \geq q \iff [p, A; 1 - p, B] \succ [q, A; 1 - q, B])\]
Rational preferences contd.

Violating the constraints leads to self-evident irrationality

For example: an agent with intransitive preferences can be induced to give away all its money

If $B \succ C$, then an agent who has $C$ would pay (say) 1 cent to get $B$

If $A \succ B$, then an agent who has $B$ would pay (say) 1 cent to get $A$

If $C \succ A$, then an agent who has $A$ would pay (say) 1 cent to get $C$
Maximizing expected utility

Theorem (Ramsey, 1931; von Neumann and Morgenstern, 1944):
Given preferences satisfying the constraints
there exists a real-valued function \( U \) such that
\[
U(A) \geq U(B) \iff A \succeq B \\
U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)
\]

MEU principle:
Choose the action that maximizes expected utility

Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities

E.g., a lookup table for perfect tic-tactoe
Utilities

Utilities map states to real numbers. Which numbers?

Standard approach to assessment of human utilities:
- compare a given state $A$ to a standard lottery $L_p$ that has
  - “best possible prize” $u_\top$ with probability $p$
  - “worst possible catastrophe” $u_\perp$ with probability $(1 - p)$
- adjust lottery probability $p$ until $A \sim L_p$

pay $\$30$ ~ $L$

\[ 0.999999 \quad \text{continue as before} \]
\[ 0.000001 \quad \text{instant death} \]
Utility scales

Normalized utilities: $u^\top = 1.0, u_\bot = 0.0$

Micromorts: one-millionth chance of death
  useful for Russian roulette, paying to reduce product risks, etc.

QALYs: quality-adjusted life years
  useful for medical decisions involving substantial risk

Note: behavior is invariant w.r.t. $+$ve linear transformation

$$U'(x) = k_1 U(x) + k_2 \text{ where } k_1 > 0$$

With deterministic prizes only (no lottery choices), only
ordinal utility can be determined, i.e., total order on prizes
Money

Money does not behave as a utility function

Given a lottery $L$ with expected monetary value $EMV(L)$, usually $U(L) < U(EMV(L))$, i.e., people are risk-averse

Utility curve: for what probability $p$ am I indifferent between a prize $x$ and a lottery $[p, $M$; (1 - $p$), $0$] for large $M$?

Typical empirical data, extrapolated with risk-prone behavior:
For each $x$, adjust $p$ until half the class votes for lottery ($M=10,000$)
Decision networks

Add action nodes and utility nodes to belief networks to enable rational decision making.

Algorithm:
For each value of action node
    compute expected value of utility node given action, evidence
Return MEU action
How can we handle utility functions of many variables $X_1 \ldots X_n$? E.g., what is $U(Deaths, Noise, Cost)$?

How can complex utility functions be assessed from preference behaviour?

Idea 1: identify conditions under which decisions can be made without complete identification of $U(x_1, \ldots, x_n)$

Idea 2: identify various types of independence in preferences and derive consequent canonical forms for $U(x_1, \ldots, x_n)$
Strict dominance

Typically define attributes such that $U$ is monotonic in each

**Strict dominance**: choice $B$ strictly dominates choice $A$ iff

$$\forall i \ X_i(B) \geq X_i(A) \quad \text{(and hence} \ U(B) \geq U(A))$$

Strict dominance seldom holds in practice

**Deterministic attributes**

**Uncertain attributes**
Stochastic dominance

Distribution \( p_1 \) stochastically dominates distribution \( p_2 \) iff
\[
\forall t \int_{-\infty}^{t} p_1(x) dx \leq \int_{-\infty}^{t} p_2(t) dt
\]

If \( U \) is monotonic in \( x \), then \( A_1 \) with outcome distribution \( p_1 \) stochastically dominates \( A_2 \) with outcome distribution \( p_2 \):
\[
\int_{-\infty}^{\infty} p_1(x) U(x) dx \geq \int_{-\infty}^{\infty} p_2(x) U(x) dx
\]

Multiattribute case: stochastic dominance on all attributes \( \Rightarrow \) optimal
Stochastic dominance contd.

Stochastic dominance can often be determined without exact distributions using **qualitative** reasoning.

E.g., construction cost increases with distance from city

\[ S_1 \text{ is closer to the city than } S_2 \]

\[ \Rightarrow \quad S_1 \text{ stochastically dominates } S_2 \text{ on cost} \]

E.g., injury increases with collision speed.

Can annotate belief networks with stochastic dominance information:

\[ X \xrightarrow{+} Y \quad (X \text{ positively influences } Y) \]

means that

For every value \( z \) of \( Y \)’s other parents \( Z \)

\[ \forall x_1, x_2 \quad x_1 \geq x_2 \Rightarrow P(Y|x_1, z) \text{ stochastically dominates } P(Y|x_2, z) \]
Label the arcs + or −
Label the arcs + or −
Label the arcs + or −
Label the arcs + or −
Label the arcs + or −
Preference structure: Deterministic

\(X_1\) and \(X_2\) preferentially independent of \(X_3\) iff preference between \(\langle x_1, x_2, x_3 \rangle\) and \(\langle x'_1, x'_2, x_3 \rangle\)
does not depend on \(x_3\)

E.g., \(\langle Noise, Cost, Safety \rangle\):
\(\langle 20,000 \text{ suffer, } $4.6 \text{ billion, } 0.06 \text{ deaths/mpm} \rangle\) vs.
\(\langle 70,000 \text{ suffer, } $4.2 \text{ billion, } 0.06 \text{ deaths/mpm} \rangle\)

**Theorem** (Leontief, 1947): if every pair of attributes is P.I. of its complement, then every subset of attributes is P.I of its complement: mutual P.I..

**Theorem** (Debreu, 1960): mutual P.I. \(\Rightarrow\) \(\exists\) additive value function:

\[
V(S) = \Sigma_i V_i(X_i(S))
\]

Hence assess \(n\) single-attribute functions; often a good approximation
Preference structure: Stochastic

Need to consider preferences over lotteries:
\( X \) is utility-independent of \( Y \) iff
preferences over lotteries in \( X \) do not depend on \( y \)

Mutual U.I.: each subset is U.I of its complement
\[ \Rightarrow \exists \text{ multiplicative utility function:} \]
\[ U = k_1 U_1 + k_2 U_2 + k_3 U_3 \]
\[ + k_1 k_2 U_1 U_2 + k_2 k_3 U_2 U_3 + k_3 k_1 U_3 U_1 \]
\[ + k_1 k_2 k_3 U_1 U_2 U_3 \]

Routine procedures and software packages for generating preference tests to identify various canonical families of utility functions
Value of information

Idea: compute value of acquiring each possible piece of evidence
Can be done **directly from decision network**

Example: buying oil drilling rights
- Two blocks $$A$$ and $$B$$, exactly one has oil, worth $$k$$
- Prior probabilities 0.5 each, mutually exclusive
- Current price of each block is $$k/2$$
- “Consultant” offers accurate survey of $$A$$. Fair price?

Solution: compute expected value of information
- $$= \text{expected value of best action given the information}
- \quad - \text{expected value of best action without information}$$
- Survey may say “oil in A” or “no oil in A”, **prob. 0.5 each** (given!)
  - $$= [0.5 \times \text{value of “buy A” given “oil in A”}
  \quad + 0.5 \times \text{value of “buy B” given “no oil in A”}]$$
  - $$- 0$$
  - $$= (0.5 \times k/2) + (0.5 \times k/2) - 0 = k/2$$
General formula

Current evidence $E$, current best action $\alpha$
Possible action outcomes $S_i$, potential new evidence $E_j$

$$EU(\alpha|E) = \max_a \sum_i U(S_i) \ P(S_i|E, a)$$

Suppose we knew $E_j = e_{jk}$, then we would choose $\alpha_{e_{jk}}$ s.t.

$$EU(\alpha_{e_{jk}}|E, E_j = e_{jk}) = \max_a \sum_i U(S_i) \ P(S_i|E, a, E_j = e_{jk})$$

$E_j$ is a random variable whose value is currently unknown
$\Rightarrow$ must compute expected gain over all possible values:

$$VPI_E(E_j) = \left( \sum_k P(E_j = e_{jk}|E) EU(\alpha_{e_{jk}}|E, E_j = e_{jk}) \right) - EU(\alpha|E)$$

(VPI = value of perfect information)
Properties of VPI

Nonnegative—in expectation, not post hoc

∀ j, E \( VPI_E(E_j) \geq 0 \)

Nonadditive—consider, e.g., obtaining \( E_j \) twice

\( VPI_E(E_j, E_k) \neq VPI_E(E_j) + VPI_E(E_k) \)

Order-independent

\( VPI_E(E_j, E_k) = VPI_E(E_j) + VPI_{E,E_j}(E_k) = VPI_E(E_k) + VPI_{E,E_k}(E_j) \)

Note: when more than one piece of evidence can be gathered, maximizing VPI for each to select one is not always optimal

⇒ evidence-gathering becomes a **sequential** decision problem
Qualitative behaviors

a) Choice is obvious, information worth little
b) Choice is nonobvious, information worth a lot
c) Choice is nonobvious, information worth little