Inference in Bayesian networks

Chapter 14.4–5

Outline

- ◇ Exact inference by enumeration
- ◇ Exact inference by variable elimination
- ◇ Approximate inference by stochastic simulation
- ◇ Approximate inference by Markov chain Monte Carlo

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

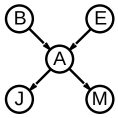
Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:



$\mathbf{P}(B|j,m)$
$= \mathbf{P}(B,j,m)/P(j,m)$
$= \alpha\mathbf{P}(B,j,m)$
$= \alpha\ \Sigma_e\ \Sigma_a\ \mathbf{P}(B,e,a,j,m)$

Rewrite full joint entries using product of CPT entries:
$\mathbf{P}(B|j,m)$
$= \alpha\ \Sigma_e\ \Sigma_a\ \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
$= \alpha\mathbf{P}(B)\ \Sigma_e\ P(e)\ \Sigma_a\ \mathbf{P}(a|B,e)P(j|a)P(m|a)$

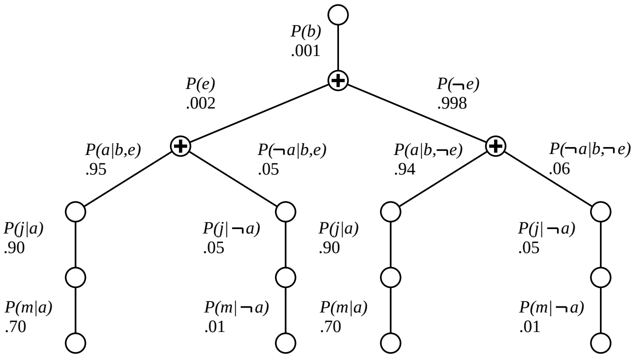
Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Enumeration algorithm

```
function Enumeration-Ask($X$, $\mathbf{e}$, $bn$) returns a distribution over $X$
   inputs: $X$, the query variable
           $\mathbf{e}$, observed values for variables $\mathbf{E}$
           $bn$, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$
   $\mathbf{Q}(X) \leftarrow$ a distribution over $X$, initially empty
   for each value $x_i$ of $X$ do
       extend $\mathbf{e}$ with value $x_i$ for $X$
       $\mathbf{Q}(x_i) \leftarrow$ Enumerate-All(Vars[$bn$], $\mathbf{e}$)
   return Normalize($\mathbf{Q}(X)$)
```

```
function Enumerate-All($vars$, $\mathbf{e}$) returns a real number
   if Empty?($vars$) then return 1.0
   $Y \leftarrow$ First($vars$)
   if $Y$ has value $y$ in $\mathbf{e}$
       then return $P(y \mid Pa(Y)) \times$ Enumerate-All(Rest($vars$), $\mathbf{e}$)
       else return $\Sigma_y P(y \mid Pa(Y)) \times$ Enumerate-All(Rest($vars$), $\mathbf{e}_y$)
           where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$
```

Evaluation tree



Enumeration is inefficient: repeated computation
e.g., computes $P(j|a)P(m|a)$ for each value of e

Inference by variable elimination

Variable elimination: carry out summations right-to-left,
storing intermediate results (factors) to avoid recomputation

$$\begin{aligned}
\mathbf{P}(B|j,m) &\\
&= \alpha \underbrace{\mathbf{P}(B)}_{B} \Sigma_e \underbrace{P(e)}_{E} \Sigma_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}\\
&= \alpha \mathbf{P}(B)\Sigma_e P(e)\Sigma_a \mathbf{P}(a|B,e) P(j|a) f_M(a)\\
&= \alpha \mathbf{P}(B)\Sigma_e P(e)\Sigma_a \mathbf{P}(a|B,e) f_J(a) f_M(a)\\
&= \alpha \mathbf{P}(B)\Sigma_e P(e)\Sigma_a f_A(a,b,e) f_J(a) f_M(a)\\
&= \alpha \mathbf{P}(B)\Sigma_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)}\\
&= \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)}\\
&= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}$$

Variable elimination: Basic operations

Summing out a variable from a product of factors:
move any constant factors outside the summation
add up submatrices in pointwise product of remaining factors

$$\Sigma_x f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \, \Sigma_x \, f_{i+1} \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f_{\bar{X}}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Pointwise product of factors f_1 and f_2:

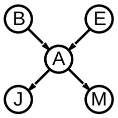
$$\begin{aligned}
&f_1(x_1,\ldots,x_j,y_1,\ldots,y_k) \times f_2(y_1,\ldots,y_k,z_1,\ldots,z_l)\\
&\quad = f(x_1,\ldots,x_j,y_1,\ldots,y_k,z_1,\ldots,z_l)
\end{aligned}$$

E.g., $f_1(a,b) \times f_2(b,c) = f(a,b,c)$

Variable elimination algorithm

```
function ELIMINATION-ASK($X$, $\mathbf{e}$, $bn$) returns a distribution over $X$
  inputs: $X$, the query variable
          $\mathbf{e}$, evidence specified as an event
          $bn$, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$
  $factors \leftarrow []$; $vars \leftarrow$ REVERSE(VARS[$bn$])
  for each $var$ in $vars$ do
      $factors \leftarrow$ [MAKE-FACTOR($var$, $\mathbf{e}$)|$factors$]
      if $var$ is a hidden variable then $factors \leftarrow$ SUM-OUT($var$, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))
```

Irrelevant variables



Consider the query $P(JohnCalls|Burglary = true)$

$P(J|b) = \alpha P(b) \Sigma_e P(e) \Sigma_a P(a|b,e) P(J|a) \Sigma_m P(m|a)$

Sum over m is identically 1; M is **irrelevant** to the query

Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$
so $MaryCalls$ is irrelevant

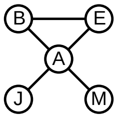
(Compare this to backward chaining from the query in Horn clause KBs)

Irrelevant variables contd.

Defn: moral graph of Bayes net: marry all parents and drop arrows

Defn: $\mathbf{A}$ is m-separated from $\mathbf{B}$ by $\mathbf{C}$ iff separated by $\mathbf{C}$ in the moral graph

Thm 2: Y is irrelevant if m-separated from X by $\mathbf{E}$



For $P(JohnCalls|Alarm = true)$, both
$Burglary$ and $Earthquake$ are irrelevant

Complexity of exact inference

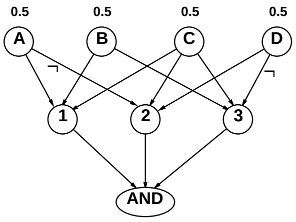
Singly connected networks (or polytrees):
- any two nodes are connected by at most one (undirected) path
- time and space cost of variable elimination are $O(d^k n)$

Multiply connected networks:
- can reduce 3SAT to exact inference ⇒ NP-hard
- equivalent to counting 3SAT models ⇒ #P-complete

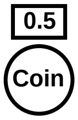
1. A v B v C

2. C v D v ¬A

3. B v C v ¬D



Inference by stochastic simulation



Basic idea:
1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P

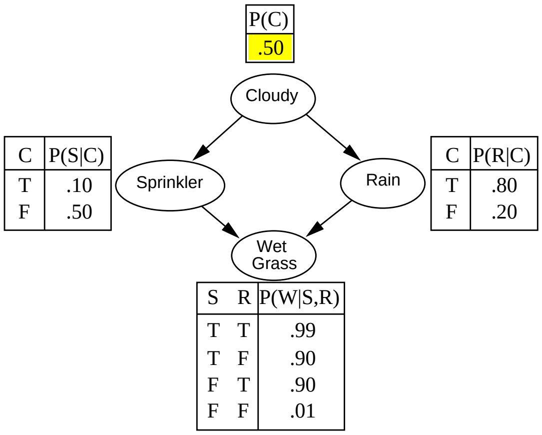
Outline:
- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an empty network

```
function PRIOR-SAMPLE($bn$) returns an event sampled from $bn$
    inputs: $bn$, a belief network specifying joint distribution $\mathbf{P}(X_1,\ldots,X_n)$

    $\mathbf{x}$ ← an event with $n$ elements
    for $i = 1$ to $n$ do
        $x_i$ ← a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
            given the values of $Parents(X_i)$ in $\mathbf{x}$
    return $\mathbf{x}$
```

Example



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T	.10
F	.50

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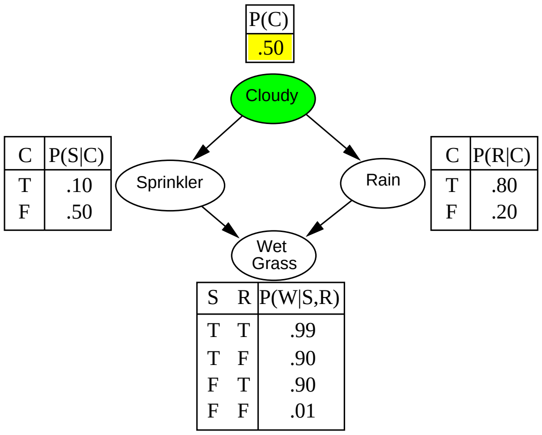

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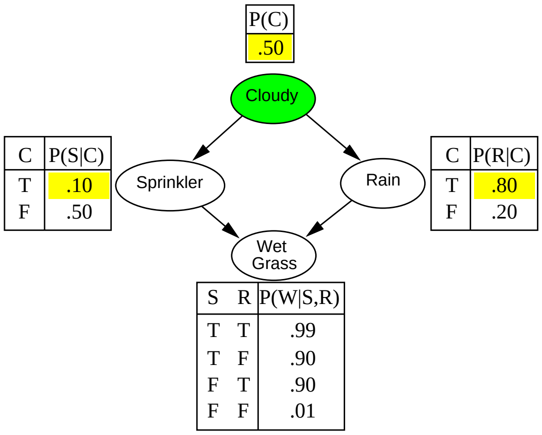

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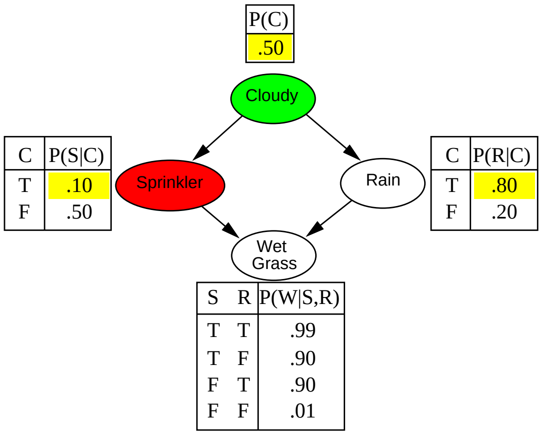

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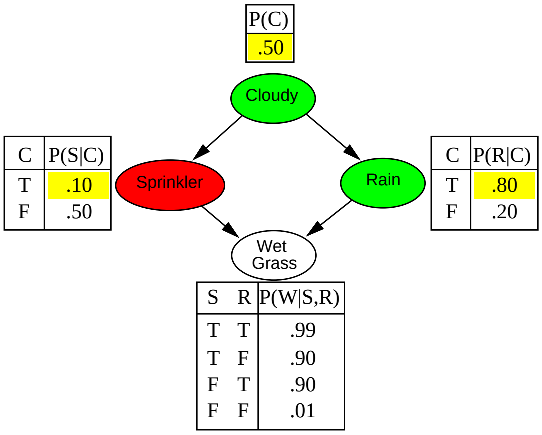

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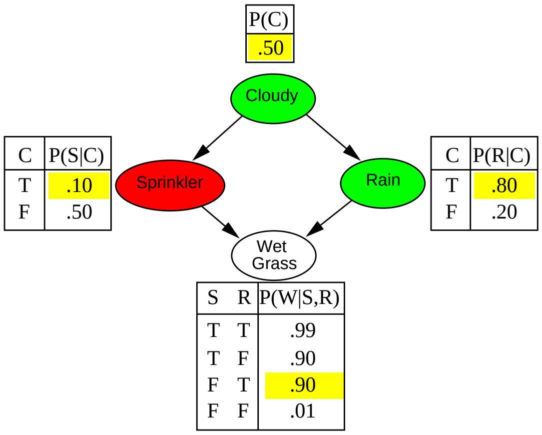

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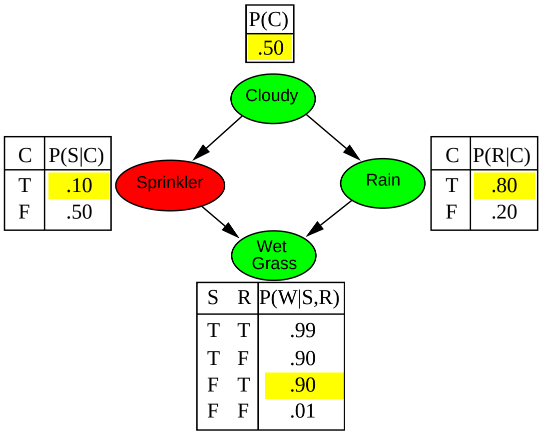

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Sampling from an empty network contd.

Probability that PRIORSAMPLE generates a particular event
$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$
i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJECTION-SAMPLING($X$, $\mathbf{e}$, $bn$, $N$) returns an estimate of $P(X|\mathbf{e})$
   local variables: $\mathbf{N}$, a vector of counts over $X$, initially zero
   for $j$ = 1 to $N$ do
       $\mathbf{x}$ ← PRIOR-SAMPLE($bn$)
       if $\mathbf{x}$ is consistent with $\mathbf{e}$ then
           $\mathbf{N}[x]$ ← $\mathbf{N}[x]$+1 where $x$ is the value of $X$ in $\mathbf{x}$
   return NORMALIZE($\mathbf{N}[X]$)
```

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples
27 samples have $Sprinkler = true$
Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19 \rangle) = \langle 0.296, 0.704 \rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e}) = \alpha \mathbf{N}_{PS}(X, \mathbf{e})$ (algorithm defn.)
$= \mathbf{N}_{PS}(X, \mathbf{e})/N_{PS}(\mathbf{e})$ (normalized by $N_{PS}(\mathbf{e})$)
$\approx \mathbf{P}(X, \mathbf{e})/P(\mathbf{e})$ (property of PRIORSAMPLE)
$= \mathbf{P}(X|\mathbf{e})$ (defn. of conditional probability)

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables,
and weight each sample by the likelihood it accords the evidence

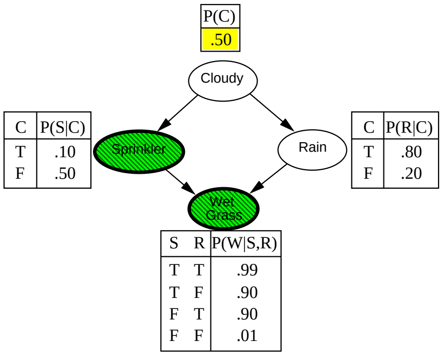
```
function LIKELIHOOD-WEIGHTING($X$, $\mathbf{e}$, $bn$, $N$) returns an estimate of $P(X|\mathbf{e})$
    local variables: $\mathbf{W}$, a vector of weighted counts over $X$, initially zero

    for $j = 1$ to $N$ do
        $\mathbf{x}$, $w \leftarrow$ WEIGHTED-SAMPLE($bn$)
        $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where $x$ is the value of $X$ in $\mathbf{x}$
    return NORMALIZE($\mathbf{W}[X]$)
```

```
function WEIGHTED-SAMPLE($bn$, $\mathbf{e}$) returns an event and a weight

    $\mathbf{x} \leftarrow$ an event with $n$ elements; $w \leftarrow 1$
    for $i = 1$ to $n$ do
        if $X_i$ has a value $x_i$ in $\mathbf{e}$
            then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$
            else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
    return $\mathbf{x}$, $w$
```

Likelihood weighting example



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$w = 1.0$

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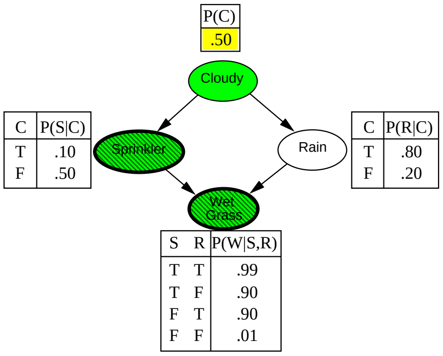

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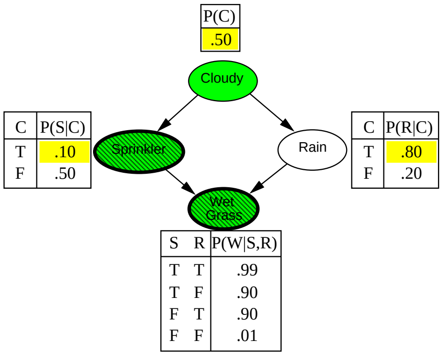

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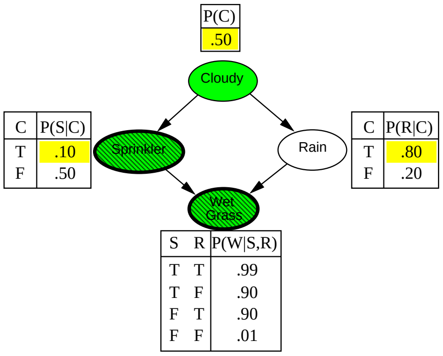

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$w = 1.0 \times 0.1$

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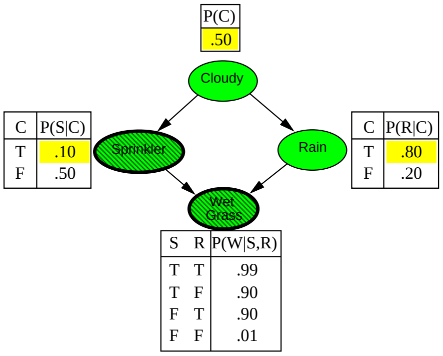

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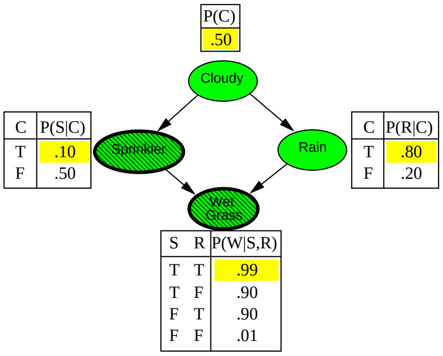

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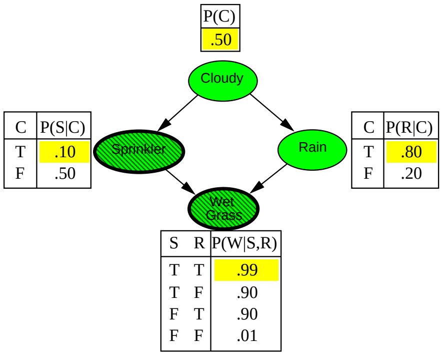
Likelihood weighting example



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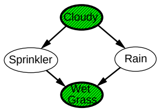
$w = 1.0 \times 0.1 \times 0.99 = 0.099$

Likelihood weighting analysis



Sampling probability for WeightedSample is
$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i|parents(Z_i))$
Note: pays attention to evidence in **ancestors** only
$\Rightarrow$ somewhere "in between" prior and posterior distribution

Weight for a given sample $\mathbf{z},\mathbf{e}$ is
$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i|parents(E_i))$

Weighted sampling probability is
$S_{WS}(\mathbf{z},\mathbf{e}) w(\mathbf{z},\mathbf{e})$
$= \prod_{i=1}^{l} P(z_i|parents(Z_i)) \;\; \prod_{i=1}^{m} P(e_i|parents(E_i))$
$= P(\mathbf{z},\mathbf{e})$ (by standard global semantics of network)

Hence likelihood weighting returns consistent estimates
but performance still degrades with many evidence variables
because a few samples have nearly all the total weight

Approximate inference using MCMC

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

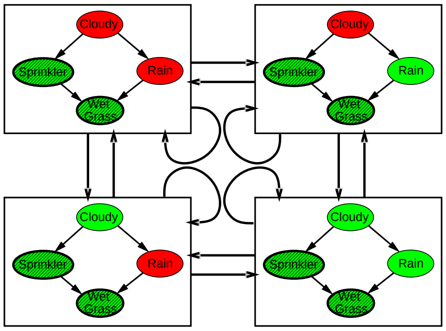
```
function MCMC-Ask($X$, $\mathbf{e}$, $bn$, $N$) returns an estimate of $P(X|\mathbf{e})$
   local variables: $\mathbf{N}[X]$, a vector of counts over $X$, initially zero
                    $\mathbf{Z}$, the nonevidence variables in $bn$
                    $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$

   initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
   for $j = 1$ to $N$ do
      for each $Z_i$ in $\mathbf{Z}$ do
         sample the value of $Z_i$ in $\mathbf{x}$ from $\mathbf{P}(Z_i|mb(Z_i))$
            given the values of $MB(Z_i)$ in $\mathbf{x}$
         $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where $x$ is the value of $X$ in $\mathbf{x}$
   return Normalize($\mathbf{N}[X]$)
```

Can also choose a variable to sample at random each time

The Markov chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC example contd.

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

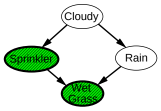
E.g., visit 100 states
31 have $Rain = true$, 69 have $Rain = false$

$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
$= \text{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$

Theorem: chain approaches stationary distribution:
long-run fraction of time spent in each state is exactly proportional to its posterior probability

Markov blanket sampling

Markov blanket of $Cloudy$ is
$Sprinkler$ and $Rain$
Markov blanket of $Rain$ is
$Cloudy$, $Sprinkler$, and $WetGrass$



Probability given the Markov blanket is calculated as follows:

$$P(x_i'|mb(X_i)) = P(x_i'|parents(X_i))\prod_{Z_j \in Children(X_i)} P(z_j|parents(Z_j))$$

Easily implemented in message-passing parallel systems, brains

Main computational problems:
1) Difficult to tell if convergence has been achieved
2) Can be wasteful if Markov blanket is large:
   $P(X_i|mb(X_i))$ won't change much (law of large numbers)

Summary

Exact inference by variable elimination:
- polytime on polytrees, NP-hard on general graphs
- space = time, very sensitive to topology

Approximate inference by LW, MCMC:
- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables