# First-order logic

## Chapter 8

# Outline

- ♢ Why FOL?
- ♢ Syntax and semantics of FOL
- ♢ Fun with sentences
- ♢ Wumpus world in FOL

# Pros and cons of propositional logic

- Propositional logic is **declarative**: pieces of syntax correspond to facts
- Propositional logic allows partial/disjunctive/negated information
  (unlike most data structures and databases)
- Propositional logic is **compositional**:
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- Meaning in propositional logic is **context-independent**
  (unlike natural language, where meaning depends on context)
- Propositional logic has very limited expressive power
  (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

# First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . .,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of . . .

# Logics in general

| Language | Ontological Commitment | Epistemological Commitment |
|---|---|---|
| Propositional logic | facts | true/false/unknown |
| First-order logic | facts, objects, relations | true/false/unknown |
| Temporal logic | facts, objects, relations, times | true/false/unknown |
| Probability theory | facts | degree of belief |
| Fuzzy logic | facts + degree of truth | known interval value |

# Syntax of FOL: Basic elements

| | |
|---|---|
| Constants | $KingJohn,\ 2,\ UCB, \ldots$ |
| Predicates | $Brother,\ >, \ldots$ |
| Functions | $Sqrt,\ LeftLegOf, \ldots$ |
| Variables | $x,\ y,\ a,\ b, \ldots$ |
| Connectives | $\wedge\ \vee\ \neg\ \Rightarrow\ \Leftrightarrow$ |
| Equality | $=$ |
| Quantifiers | $\forall\ \exists$ |

# Atomic sentences

$$\text{Atomic sentence} = predicate(term_1, \ldots, term_n)$$
$$\text{or } term_1 = term_2$$

$$\text{Term} = function(term_1, \ldots, term_n)$$
$$\text{or } constant \text{ or } variable$$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

# Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg>(1,2)$

# Truth in first-order logic

Sentences are true with respect to a model and an interpretation

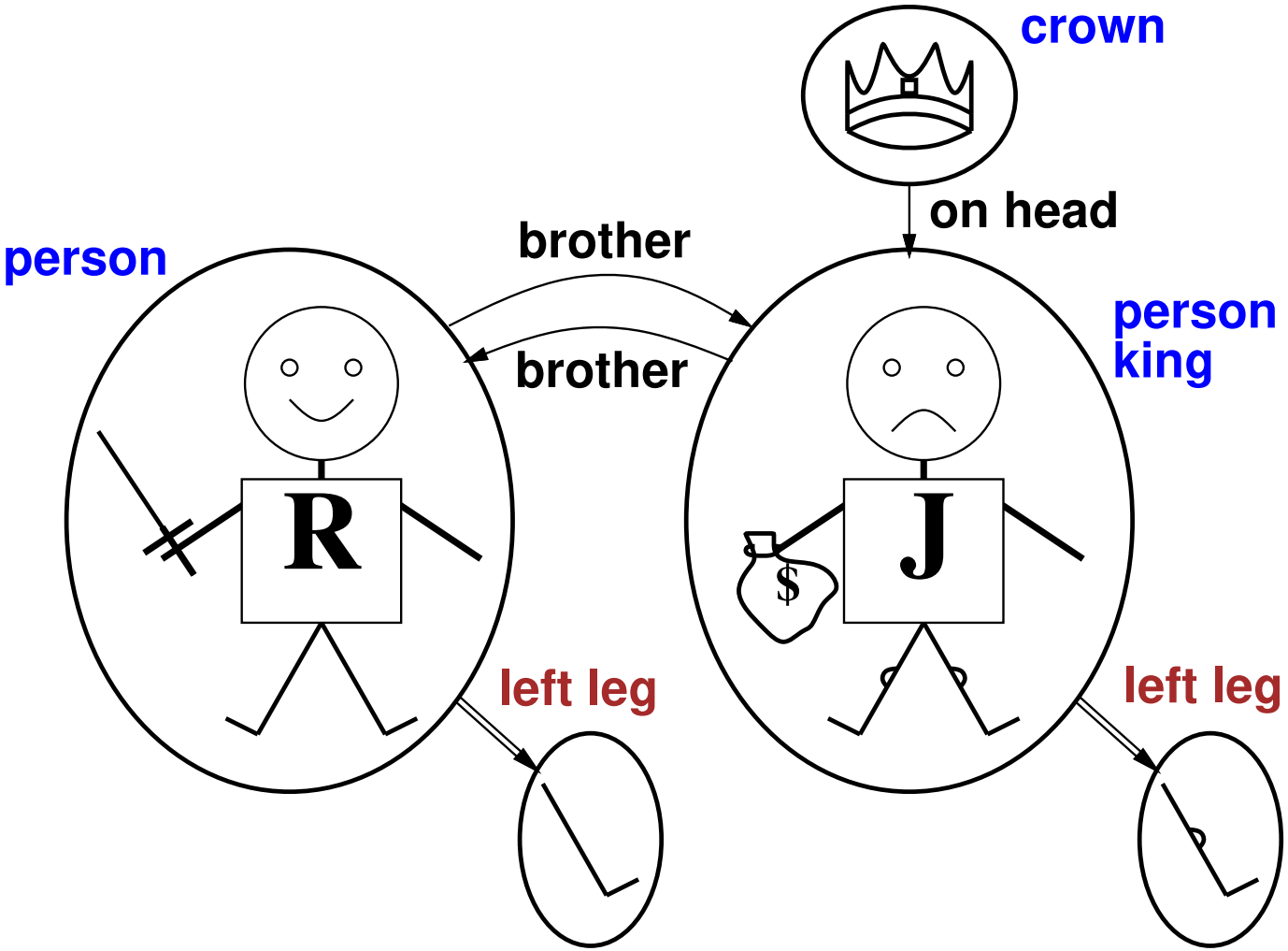
Model contains $\geq 1$ objects (domain elements) and relations among them

Interpretation specifies referents for

- constant symbols $\rightarrow$ objects
- predicate symbols $\rightarrow$ relations
- function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

# Models for FOL: Example



# Truth example

Consider the interpretation in which
$Richard \rightarrow$ Richard the Lionheart
$John \rightarrow$ the evil King John
$Brother \rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

# Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements $n$ from $1$ to $\infty$
    For each $k$-ary predicate $P_k$ in the vocabulary
        For each possible $k$-ary relation on $n$ objects
            For each constant symbol $C$ in the vocabulary
                For each choice of referent for $C$ from $n$ objects . . .

Computing entailment by enumerating FOL models is not easy!

# Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \ At(x, Berkeley) \Rightarrow Smart(x)$

$\forall x \ P$ is true in a model $m$ iff $P$ is true with $x$ being **each** possible object in the model

**Roughly** speaking, equivalent to the conjunction of instantiations of $P$

$$
\begin{aligned}
& (At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)) \\
\wedge \ & (At(Richard, Berkeley) \Rightarrow Smart(Richard)) \\
\wedge \ & (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)) \\
\wedge \ & \ldots
\end{aligned}
$$

# A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \;\; At(x, Berkeley) \wedge Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

# Existential quantification

$\exists \langle variables \rangle \ \ \langle sentence \rangle$

Someone at Stanford is smart:
$\exists x \ \ At(x, Stanford) \wedge Smart(x)$

$\exists x \ \ P$ is true in a model $m$ iff $P$ is true with $x$ being **some** possible object in the model

**Roughly** speaking, equivalent to the disjunction of instantiations of $P$

$$
\begin{array}{ll}
 & (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\
\vee & (At(Richard, Stanford) \wedge Smart(Richard)) \\
\vee & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\
\vee & \ldots
\end{array}
$$

# Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \;\; At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!

# Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$ (why??)

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$ (why??)

$\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$

$\exists x \ \forall y \ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y \ \exists x \ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x \ Likes(x, IceCream) \qquad \neg\exists x \ \neg Likes(x, IceCream)$

$\exists x \ Likes(x, Broccoli) \qquad \neg\forall x \ \neg Likes(x, Broccoli)$

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$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

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One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$.

A first cousin is a child of a parent's sibling

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$\forall x, y \;\; Sibling(x, y) \Leftrightarrow Sibling(y, x)$.

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$\forall x, y \;\; Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$.

A first cousin is a child of a parent's sibling

$\forall x, y \;\; FirstCousin(x, y) \Leftrightarrow \exists p, ps \;\; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

# Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists m, f \;\; \neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

# Open and Closed Worlds

Suppose the KB contains the following facts:

$Teaches(Russell, CS188, Spring05)$ $\quad$ $Teaches(Russell, CS298\text{-}10, Spring05)$

How many courses does Prof. Russell teach in Spring 2005???

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Database system: $2$

First-order logic: between $1$ and $\infty$

Database systems assume **unique names** and **closed world**

# Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists\, a \;\; Action(a, 5))$

I.e., does $KB$ entail any particular actions at $t = 5$?

Answer: $Yes, \; \{a/Shoot\}$ $\leftarrow$ substitution (binding list)

Given a sentence $S$ and a substitution $\sigma$,
$S\sigma$ denotes the result of plugging $\sigma$ into $S$; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all $\sigma$ such that $KB \models S\sigma$

# Knowledge base for the wumpus world

"Perception"

$\forall\, b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$

$\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall\, t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall\, t \;\; AtGold(t) \land \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed

$\Rightarrow$ keeping track of change is essential

# Deducing hidden properties

Properties of locations:

$\forall\, x, t \;\; At(Agent, x, t) \land Smelt(t) \;\Rightarrow\; Smelly(x)$

$\forall\, x, t \;\; At(Agent, x, t) \land Breeze(t) \;\Rightarrow\; Breezy(x)$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall\, y \;\; Breezy(y) \;\Rightarrow\; \exists\, x \;\; Pit(x) \land Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall\, x, y \;\; Pit(x) \land Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:

$$\forall\, y \;\; Breezy(y) \;\Leftrightarrow\; [\exists\, x \;\; Pit(x) \land Adjacent(x, y)]$$

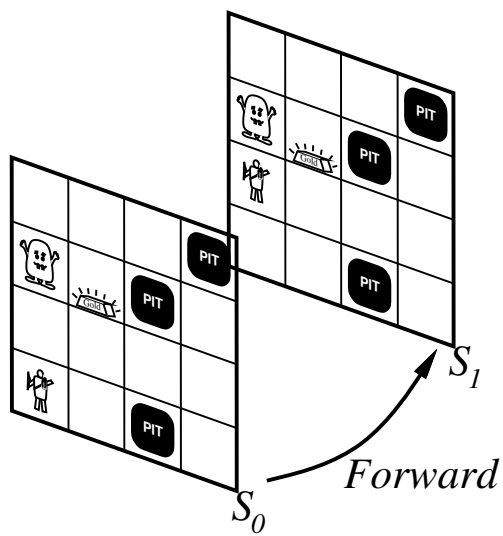
# Keeping track of change

Facts hold in situations, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:

Adds a situation argument to each non-eternal predicate
E.g., $Now$ in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing $a$ in $s$



# Describing actions I

"Effect" axiom—describe changes due to action

$\forall s \;\; AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe **non-changes** due to action

$\forall s \;\; HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$

Frame problem: find an elegant way to handle non-change

(a) representation—avoid frame axioms

(b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

# Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

P true afterwards $\Leftrightarrow$ [an action made P true
$\vee$ P true already and no action made P false]

For holding the gold:

$$\forall a, s \;\; Holding(Gold, Result(a, s)) \Leftrightarrow$$
$$[(a = Grab \wedge AtGold(s))$$
$$\vee (Holding(Gold, s) \wedge a \neq Release)]$$

# Making plans

Initial condition in KB:

$At(Agent, [1,1], S_0)$
$At(Gold, [1,2], S_0)$

Query: $Ask(KB, \exists\, s \;\; Holding(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at $S_0$ and that $S_0$ is the only situation described in the KB

# Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing $p$ in $s$

Then the query $Ask(KB, \exists\, p \;\; Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:

$\forall\, s \;\; PlanResult([\,], s) = s$

$\forall\, a, p, s \;\; PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

# Summary

First-order logic:

- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:

- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB