First-order logic

Chapter 8

Outline

◇ Why FOL?

◇ Syntax and semantics of FOL

◇ Fun with sentences

◇ Wumpus world in FOL

Pros and cons of propositional logic

☺ Propositional logic is **declarative**: pieces of syntax correspond to facts

☺ Propositional logic allows partial/disjunctive/negated information
(unlike most data structures and databases)

☺ Propositional logic is **compositional**:
meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

☺ Meaning in propositional logic is **context-independent**
(unlike natural language, where meaning depends on context)

☹ Propositional logic has very limited expressive power
(unlike natural language)
E.g., cannot say "pits cause breezes in adjacent squares"
except by writing one sentence for each square

First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . ., brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of . . .

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic elements

Constants	$KingJohn,\ 2,\ UCB, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Rightarrow\ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic sentences

Atomic sentence $= predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $= function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg>(1,2)$

Truth in first-order logic

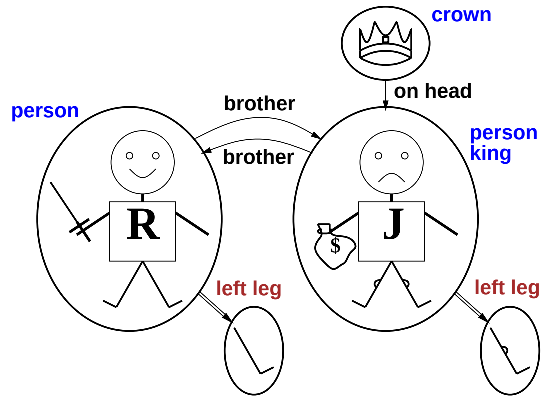
Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them

Interpretation specifies referents for
constant symbols → objects
predicate symbols → relations
function symbols → functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example



Truth example

Consider the interpretation in which
$Richard$ → Richard the Lionheart
$John$ → the evil King John
$Brother$ → the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true
just in case Richard the Lionheart and the evil King John
are in the brotherhood relation in the model

Mathematically, the brotherhood relation = set of 2-tuples of objects:

. . .
⟨Richard the Lionheart, the evil King John⟩
⟨the evil King John, Richard the Lionheart⟩
. . .
⟨Tweedledum, Tweedledee⟩
⟨Tweedledee, Tweedledum⟩
. . .

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
For each k-ary predicate P_k in the vocabulary
For each possible k-ary relation on n objects
For each constant symbol C in the vocabulary
For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \; \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \;\; At(x, Berkeley) \;\Rightarrow\; Smart(x)$

$\forall x \;\; P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Berkeley) \;\Rightarrow\; Smart(KingJohn)) \\
\wedge\; & (At(Richard, Berkeley) \;\Rightarrow\; Smart(Richard)) \\
\wedge\; & (At(Berkeley, Berkeley) \;\Rightarrow\; Smart(Berkeley)) \\
\wedge\; & \ldots
\end{aligned}
$$

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$\forall x \;\; At(x, Berkeley) \wedge Smart(x)$

means "Everyone is at Berkeley and everyone is smart"

Existential quantification

$\exists \langle variables \rangle \; \langle sentence \rangle$

Someone at Stanford is smart:
$\exists x \;\; At(x, Stanford) \wedge Smart(x)$

$\exists x \;\; P$ is true in a model m iff P is true with x being **some** possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\
\vee\; & (At(Richard, Stanford) \wedge Smart(Richard)) \\
\vee\; & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\
\vee\; & \ldots
\end{aligned}
$$

Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$\exists x \;\; At(x, Stanford) \;\Rightarrow\; Smart(x)$

is true if there is anyone who is not at Stanford!

Properties of quantifiers

$\forall x \;\; \forall y$ is the same as $\forall y \;\; \forall x$ (why??)

$\exists x \;\; \exists y$ is the same as $\exists y \;\; \exists x$ (why??)

$\exists x \;\; \forall y$ is **not** the same as $\forall y \;\; \exists x$

$\exists x \;\; \forall y \;\; Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y \;\; \exists x \;\; Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x \;\; Likes(x, IceCream) \qquad \neg\exists x \;\; \neg Likes(x, IceCream)$

$\exists x \;\; Likes(x, Broccoli) \qquad \neg\forall x \;\; \neg Likes(x, Broccoli)$

Fun with sentences

Brothers are siblings

Fun with sentences

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$\forall x, y \;\; Brother(x,y) \;\Rightarrow\; Sibling(x,y).$

"Sibling" is symmetric

Fun with sentences

Brothers are siblings

$\forall x, y \;\; Brother(x,y) \;\Rightarrow\; Sibling(x,y).$

"Sibling" is symmetric

$\forall x, y \;\; Sibling(x,y) \;\Leftrightarrow\; Sibling(y,x).$

One's mother is one's female parent

Fun with sentences

Brothers are siblings

$\forall x, y \;\; Brother(x,y) \;\Rightarrow\; Sibling(x,y).$

"Sibling" is symmetric

$\forall x, y \;\; Sibling(x,y) \;\Leftrightarrow\; Sibling(y,x).$

One's mother is one's female parent

$\forall x, y \;\; Mother(x,y) \;\Leftrightarrow\; (Female(x) \land Parent(x,y)).$

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$\forall x, y \;\; Brother(x,y) \;\Rightarrow\; Sibling(x,y).$

"Sibling" is symmetric

$\forall x, y \;\; Sibling(x,y) \;\Leftrightarrow\; Sibling(y,x).$

One's mother is one's female parent

$\forall x, y \;\; Mother(x,y) \;\Leftrightarrow\; (Female(x) \land Parent(x,y)).$

A first cousin is a child of a parent's sibling

$\forall x, y \;\; FirstCousin(x,y) \;\Leftrightarrow\; \exists p, ps \;\; Parent(p,x) \land Sibling(ps,p) \land Parent(ps,y)$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall x, y \;\; Sibling(x,y) \;\Leftrightarrow\; [\neg(x = y) \land \exists m, f \;\; \neg(m = f) \land Parent(m,x) \land Parent(f,x) \land Parent(m,y) \land Parent(f,y)]$$

Open and Closed Worlds

Suppose the KB contains the following facts:

$Teaches(Russell, CS188, Fall05) \qquad Teaches(Russell, CS294\text{-}10, Fall05)$

How many courses does Prof. Russell teach in Fall 2005???

Open and Closed Worlds

Suppose the KB contains the following facts:

$Teaches(Russell, CS188, Fall05)$ $Teaches(Russell, CS294\text{-}10, Fall05)$

How many courses does Prof. Russell teach in Fall 2005???

Database system: 2

First-order logic: between 1 and ∞

Database systems assume **unique names** and **closed world**

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists\, a\;\; Action(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

Answer: $Yes,\ \{a/Shoot\}$ ← substitution (binding list)

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

"Perception"
$\forall\, b, g, t\;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$
$\forall\, s, b, t\;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall\, t\;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?
$\forall\, t\;\; AtGold(t) \land \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed
⇒ keeping track of hidden state is essential

Deducing hidden properties

Properties of locations:
$\forall\, x, t\;\; At(Agent, x, t) \land Smelt(t) \;\Rightarrow\; Smelly(x)$
$\forall\, x, t\;\; At(Agent, x, t) \land Breeze(t) \;\Rightarrow\; Breezy(x)$

Squares are breezy near a pit:

Definition for the $Breezy$ predicate:
$\forall\, y\;\; Breezy(y) \;\Leftrightarrow\; [\exists\, x\;\; Pit(x) \land Adjacent(x, y)]$

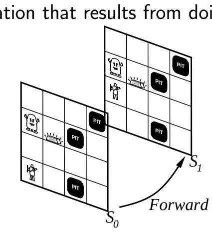
Note that one sentence suffices to cover all squares and times

Keeping track of change

Facts hold in situations, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:
Adds a situation argument to each non-eternal predicate
E.g., Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s



Describing actions I

"Effect" axiom—describe changes due to action
$\forall\, s\;\; AtGold(s) \;\Rightarrow\; Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe **non-changes** due to action
$\forall\, s\;\; HaveArrow(s) \;\Rightarrow\; HaveArrow(Result(Grab, s))$

Frame problem: find an elegant way to handle non-change
(a) representation—avoid frame axioms
(b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

P true afterwards	$\Leftrightarrow$	[an action made P true
	$\vee$	P true already and no action made P false]

For holding the gold:

$\forall a, s \;\; Holding(Gold, Result(a, s)) \;\Leftrightarrow$
$[(a = Grab \wedge AtGold(s))$
$\vee (Holding(Gold, s) \wedge a \neq Release)]$

Making plans

Initial condition in KB:

$At(Agent, [1, 1], S_0)$
$At(Gold, [1, 2], S_0)$

Query: $Ask(KB, \exists\, s \;\; Holding(Gold, s))$
i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$
i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists\, p \;\; Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:

$\forall\, s \;\; PlanResult([\,], s) = s$
$\forall\, a, p, s \;\; PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

First-order logic:
- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB