1 Utility function estimation

1.1 Problem 1

Suppose we know that a user has a Cobb-Douglas utility function, i.e. their utility function is of the form:
\[ u(x, y) = x^\alpha y^{1-\alpha} \]

Here, \( \alpha \in [0, 1] \) is an unknown value.

Now, suppose we make a sequence of observations where the user chooses the bundle \((x_i^+, y_i^+)\) over \((x_i, y_i)\), for observation indices \(i = 1, \ldots, n\).

Derive the inequality conditions that arise from this.

1.2 Problem 2

Similarly to Problem 1, suppose the user has:
\[ u(x) = \prod_{i=1}^I x_i^{\alpha_i} \]

Here, \( \alpha \in \mathbb{R}^I \) is an unknown vector that sums to 1, i.e. \( \sum_{i=1}^I \alpha_i = 1 \).

Additionally, suppose we know the user’s budget \(I\), and the price vector \(p = (p_1, \ldots, p_I)\) for the goods. Suppose the user wants to maximize their utility subject to their budget constraint. Write down this optimization and derive optimality conditions for it. If we observe one consumption decision, i.e. choice of \(x\), and we assume they are optimizing this utility function, can we calculate the utility function’s parameters, i.e. \( \alpha \)?

(We can suppose \(x_i\) can take negative values, i.e. the user can sell things if his valuation is below the market price. This means there is no \(x \geq 0\) constraint. You can also assume that the user will use the entirety of their budget on goods, i.e. \(p^Tx = I\).)

Hint: Note that whenever we want to solve \(\max f(x)\), we can equivalently always just solve \(\min -f(x)\). Additionally, we can often solve \(\max \log(f(x))\) instead of \(\max f(x)\) when this is more simple.

1.3 Solution

1) Note that \((\log u)(x, y) = \alpha \log x + (1 - \alpha) \log y\), and then our constraints become:
\[ \alpha \log x_i^+ + (1 - \alpha) \log y_i^+ \geq \alpha \log x_i + (1 - \alpha) \log y_i \]
\[ \log \frac{y_i}{y_i^+} + \alpha \left( \log \frac{x_i}{x_i^+} - \log \frac{y_i}{y_i^+} \right) \leq 0 \]

2) \[
\begin{align*}
\max_x & \quad u(x) \\
\text{subject to} & \quad p^T x \leq I \\
\min_x & \quad - (\log u)(x) \\
\text{subject to} & \quad p^T x = I
\end{align*}
\]
The first-order necessary conditions:

\[-\nabla (\log u)(x) + \lambda p = 0\]

\[\frac{\alpha_i}{x_i} = \lambda p_i\]

\[x_i = \frac{\alpha_i}{p_i \lambda}\]

Thus, we can write \(x_i\) at optimum as a function of \(\alpha\), \(p\), and \(\lambda\), where \(p\) is the only thing we know so far.

We also have \(p^T x = I\), which yields:

\[\sum_i p_i \frac{\alpha_i}{p_i \lambda} = I\]

\[\frac{1}{T} \sum_i \alpha_i = \lambda\]

Since we assumed, by the form of the Cobb-Douglas function, that \(\sum_i \alpha_i = 1\), then we can see that \(\lambda = 1/I\).

Thus:

\[\alpha_i = x_i \frac{p_i}{T}\]

We can learn this utility from data! Note that, in practice, assuming users have a Cobb-Douglas utility function and are choosing exactly the optimum is a strong assumption, and some of the readings we covered in course talk about how to ‘loosen’ these assumptions to allow users some wiggle room in their optimizations.

2 Positive homogeneity

2.1 Problem

A function \(f : \mathbb{R}^n \to \mathbb{R}^m\) is positively homogeneous of degree 1 if \(f(\alpha x) = \alpha f(x)\) for any \(x \in \mathbb{R}^n\) and \(\alpha > 0\).

Show that the directional derivative is positively homogeneous of degree 1 in \(y\). (In other words, fix \(x\) and show that the function \(f'(x; \cdot)\) is positively homogeneous of degree 1.)

Recall:

\[f'(x; y) = \lim_{\lambda \to 0} \frac{f(x + \lambda y) - f(x)}{\lambda}\]

2.2 Solution

Take \(\lambda' = \lambda \alpha\):

\[f'(x; \alpha y) = \lim_{\lambda \to 0} \frac{f(x + \lambda \alpha y) - f(x)}{\lambda} = \lim_{\lambda' \to 0} f(x + \lambda' y) - f(x) = \alpha \lim_{\lambda' \to 0} \frac{f(x + \lambda' y) - f(x)}{\lambda'} = \alpha f'(x; y)\]