Convex Relaxations for Constraint Satisfaction Problems

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Constraint Satisfaction Problem

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Maximize weighted sum of satisfied constraints
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**Examples:**
- SAT: Constraints of the form \( x_1 \lor \bar{x}_4 \lor x_7, x_i \in \{0, 1\} \)
- Max-Cut
LP Relaxation

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We impose two consistency conditions

\[ \sum_{\ell \in D} \mu_i[\ell] = 1, \quad \sum_L \lambda_j[L] = 1 \Rightarrow \text{exactly one indicator is non-zero.} \]
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- \( \sum_{\ell \in D} \mu_i[\ell] = 1, \sum_{L} \lambda_j[L] = 1 \Rightarrow \text{exactly one indicator is non-zero.} \)

- \( \mu_i[\ell] = \sum_{L(i) = \ell} \lambda_j[L] \Rightarrow \text{Consistency in assignments} \)
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2. \( \mu_i[\ell] = \sum_{L(i)=\ell} \lambda_j[L] \Rightarrow \text{Consistency in assignments} \)

Relax \( \mu_i[\ell] \in \{0, 1\} \rightarrow \mu_i[\ell] \in [0, 1] \) and \( \lambda_j[L] \in \{0, 1\} \rightarrow \lambda_j[L] \in [0, 1] \).
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Notice \( \lambda_j, \mu_i \) are \( \geq 0 \) and sum to 1. Think “probability distributions”!
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(LMI) \quad X &\succeq 0 \quad (\text{Affine}) \quad X_{(i,\ell),(i',\ell')} = 0 \quad \forall \ell \neq \ell' \\
\text{and (Affine)} \quad &\sum_{L \in \mathcal{L}_j \land L(i) = \ell \land L(i') = \ell'} \lambda_j[L] = X_{(i,\ell),(i',\ell')} 
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Solve this SDP to get covariance matrix $X$. Now how do we use it?
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### SDP Rounding

The random variables are no longer independent and hence we need to generate "dependent" set of random numbers whose covariance matrix is related to $X$. 