0.1 Triangulation-based Target Localization

In this section, we consider the problem of determining a target's absolute position based on distance or range measurements from three or more spatially-distributed sensors. Initially, we will restrict our analysis to two dimensions using a static target that is simultaneosly detected by three sensors or more sensors. We will then consider the dynamic case in which the target position changes over time and the detection times are partially overlapping. We will ground our examples using a micropower impulse radar or MIR sensor. The approach outlined here can be generalized easily to three dimensions and is applicable to any type of ranging device including sonar, ultrasonic, infrared, and general purpose radar.

Each MIR sensor can range [1] a target within its detection radius. A single range measurement, r, restricts the target's location to a circle of radius r centered at the MIR sensor. Similarly, if a second MIR sensor can provide a range measurement as well, then the target's position is restricted to one of the two points where the two circles intersect, unless of course the circles do not intersect or the target falls on the line connecting the two sensors. If a third sensor can provide a range measurement, then the target's position can be narrowed to just one position. With only two dimensions, three range estimates will exactly determine the target's location and four or more range estimates will overdetermine the solution.

If we denote the unknown location of the target as (x, y), and the i-th sensor's location as (x_i, y_i) and range estimate as r_i, then the following set of equations will hold true $\forall i$, assuming no range error

[1] To determine the distance from the sensor to a target

$$r_i = \sqrt{(x_i - x)^2 + (y_i - y)^2} \tag{0.1}$$

However, all real measurements have some degree of error, so we add an error term, Δr_i, to the range estimate to account for any errors between the actual range and the estimated range, which gives

$$r_i = \sqrt{(x_i - x)^2 + (y_i - y)^2} + \Delta r_i \tag{0.2}$$

Ignoring the error term for the moment, squaring both sides, and writing in vector notation for n independent range estimates gives

$$\begin{bmatrix} (x_1 - x)^2 + (y_1 - y)^2 \\ (x_2 - x)^2 + (y_2 - y)^2 \\ \vdots \\ (x_{n-1} - x)^2 + (y_{n-1} - y)^2 \\ (x_n - x)^2 + (y_n - y)^2 \end{bmatrix} = \begin{bmatrix} r_1^2 \\ r_2^2 \\ \vdots \\ r_{n-1}^2 \\ r_n^2 \end{bmatrix} \tag{0.3}$$

Expanding the elements on the left gives

$$\begin{bmatrix} x_1^2 - 2x_1x + x^2 + y_1^2 - 2y_1x + y^2 \\ x_2^2 - 2x_2x + x^2 + y_2^2 - 2y_2x + y^2 \\ \vdots \\ x_{n-1}^2 - 2x_{n-1}x + x^2 + y_{n-1}^2 - 2y_{n-1}x + y^2 \\ x_n^2 - 2x_nx + x^2 + y_n^2 - 2y_nx + y^2 \end{bmatrix} = \begin{bmatrix} r_1^2 \\ r_2^2 \\ \vdots \\ r_{n-1}^2 \\ r_n^2 \end{bmatrix} \tag{0.4}$$

We can linearize these equations by subtracting the bottom row from each of the remaining rows (which eliminates all unknown square terms), moving all remaining square terms (which are known) to the righthand side, and factoring the unknown variables, resulting in

$$\begin{bmatrix} 2x(x_n - x_1) + 2y(y_n - y_1) \\ 2x(x_n - x_2) + 2y(y_n - y_2) \\ \vdots \\ 2x(x_n - x_{n-1}) + 2y(y_n - y_{n-1}) \end{bmatrix} = \begin{bmatrix} r_1^2 - r_n^2 - x_1^2 + x_n^2 - y_1^2 + y_n^2 \\ r_2^2 - r_n^2 - x_2^2 + x_n^2 - y_2^2 + y_n^2 \\ \vdots \\ r_{n-1}^2 - r_n^2 - x_{n-1}^2 + x_n^2 - y_{n-1}^2 + y_n^2 \end{bmatrix} \tag{0.5}$$

Which can be written in matrix form as

$$2\begin{bmatrix} (x_n - x_1) & (y_n - y_1) \\ (x_n - x_2) & (y_n - y_2) \\ \vdots & \vdots \\ (x_n - x_{n-1}) & (y_n - y_{n-1}) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1^2 - r_n^2 - x_1^2 + x_n^2 - y_1^2 + y_n^2 \\ r_2^2 - r_n^2 - x_2^2 + x_n^2 - y_2^2 + y_n^2 \\ \vdots \\ r_{n-1}^2 - r_n^2 - x_{n-1}^2 + x_n^2 - y_{n-1}^2 + y_n^2 \end{bmatrix} \tag{0.6}$$

For an exactly determined solution where precisely three independent sensors report estimated ranges, we can write

$$2\begin{bmatrix} (x_n - x_1) & (y_n - y_1) \\ (x_n - x_2) & (y_n - y_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1^2 - r_n^2 - x_1^2 + x_n^2 - y_1^2 + y_n^2 \\ r_2^2 - r_n^2 - x_2^2 + x_n^2 - y_2^2 + y_n^2 \end{bmatrix} \tag{0.7}$$

which is easily solvable as a system of two equations in two unknowns.

If we have more than three sensors report their range, then the solution is overdetermined. In such a case, we can either eliminate the extraneous equation(s) or we can solve with all of the equations using least squares and the pseudoinverse.[??] The advantage of using the least squares and pseudoinverse is that when the range estimates have unknown errors, that is they are not exact, this technique provides the least squares solution that will find the "best" estimate.

The general form of n equations in m unknowns, $z_1, z_2, \ldots, z_m$ is given by

$$\mathbf{A}\mathbf{z} = \mathbf{r} \tag{0.8}$$

When $m = n$ and the determinant of $\mathbf{A} \neq 0$, the solution to the set of equations is

$$\mathbf{z} = \mathbf{A^{-1}r} \tag{0.9}$$

When $m < n$, the set of equations is overdetermined, and the solution is given by

$$\mathbf{z} = \mathbf{A}^{\#}\mathbf{r} \tag{0.10}$$

where $\mathbf{A}^{\#}$ is called the pseudoinverse of $\mathbf{A}$. $\mathbf{A}^{\#}$ exists whenever $\mathbf{A}^T\mathbf{A}$ has an inverse and is given by

$$\mathbf{A}^{\#} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T \tag{0.11}$$

Recall that the norm of a vector $\mathbf{v}$ is the inner product of the vector with itself

$$\|\mathbf{v}\| = \langle\mathbf{v}|\mathbf{v}\rangle = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2} \tag{0.12}$$

The least squares and pseudoinverse approach provides the solution that minimizes the norm $\|\mathbf{Az} - \mathbf{r}\|$. That is, this approach minimizes the distance, or error, between the two vectors $\mathbf{Az}$ and $\mathbf{r}$ as follows

$$distance(\mathbf{Az}, \mathbf{r}) = \|\mathbf{Az} - \mathbf{r}\| = \left[\sum_{i=1}^{n}((\mathbf{Az})_\mathbf{i} - \mathbf{r_i})^2\right]^{1/2} \tag{0.13}$$

which is the usual least squares solution.