1. [5 pts total] The structure on the left below consists of a rigid body attached to the end of a beam of length L, width a, and thickness b. The goal is that a vertical force $F_y$ generates only deflection in the y direction, and no rotation θ at the tip of the beam. The force acts at a distance r from the end of the beam.
   a. [2 pts] Write an expression for the rotation of the tip of the beam as a function of the moment arm r.

See W4L2 for in-depth explanation

$$\theta = y'(L) = \frac{1}{EI}\left(M_0 L + \frac{FL^2}{2}\right) = \frac{1}{EI}\left(FrL + \frac{FL^2}{2}\right)$$

   b. [1 pts] Solve for the value of r that sets the tip rotation to 0.

$$\theta = y'(L) = \frac{1}{EI}\left(M_0 L + \frac{FL^2}{2}\right) = \frac{1}{EI}\left(FrL + \frac{FL^2}{2}\right) = 0$$

$$r = -\frac{L}{2}$$

   c. [2 pts] Compare the stiffness of the mechanism in part b to the simple beam (i.e. $F_y$ applied at r=0).

$$k_{y-simple} = \frac{Eba^3}{4L^3}$$

$$k_{y-no\ rotation} = \frac{Eba^3}{L^3}$$ Since no rotation →guided end condition

$$\frac{k_{y-no\ rotation}}{k_{y-simple}} = 4$$

2. [5 pts] Repeat the previous problem, but with the goal of getting zero tip deflection.
   a. [2 pts] Expression for deflection of the tip

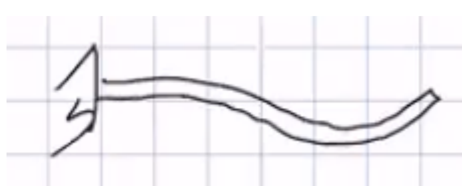
See W4L2 for in-depth explanation

$$y(L) = \frac{1}{EI}\left(\frac{M_0 L^2}{2} + \frac{FL^3}{3}\right) = \frac{1}{EI}\left(\frac{FrL^2}{2} + \frac{FL^3}{3}\right) = 0$$

   b. [1 pts] Find r that sets tip deflection to 0

$$y(L) = 0 \rightarrow r = -\frac{2L}{3}$$

   c. [2 pts. 1 pt each: tip at 0 deflection, rest of beam below] Sketch the shape of the beam under load.



3. [11 pts] In the structure on the right below, the two beams both have a width a and thickness b.

   a. [6 pts] Choose $L_2$ such that the spring constants in the x and y directions are equal.

See W4L3 2020 lecture for in-depth explanation

Same spring constant in x and y directions $\rightarrow \frac{F_x}{\Delta x} = \frac{F_y}{\Delta y}$

$k_y = \frac{3EI}{{L_1}^3}$ (negligible axial bending of $L_2$)

$k_x =$ (normal transverse spring constant for beam 2) in series with (the theta spring constant from beam 1) times $L_2$

Apply $F_x \rightarrow M_0 = F_x L_2$ (Pure Force so no $additional\ M_0\ applied$) $\rightarrow\ \theta = y_{beam1}'(L) = \frac{F_x L_2}{EI} L_1$ and the deflection is $\theta L_2$

$$k_{theta} = \frac{EI}{{L_2}^2 L_1}$$

$$k_{transverse\ beam2} = \frac{3EI}{{L_2}^3}$$

$$k_x = \frac{1}{k_{theta} + k_{transverse\ beam2}} = \frac{1}{\frac{1}{\frac{EI}{{L_2}^2 L_1}} + \frac{1}{\frac{3EI}{{L_2}^3}}} = \frac{EI}{{L_2}^2 L_1 + \frac{1}{3} {L_2}^3}$$

Set $k_y = k_x$ $and\ solve\ for\ the\ relationship\ between\ L_1\ and\ L_2$ yields

$$L_1^3 = 3\, L_2^2 L_1 + L_2^3$$

$$(L_1 / L_2)^3 = 3(L_1 / L_2) + 1$$

$(L_1 / L_2) \sim= 1.9$ so L2 should be a little over half as long as L1

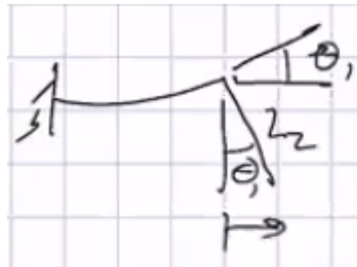
1 pt. each for ky, ktheta, ktransverse, the moment on beam 1, the deflection due to that moment, and the overall kx spring constant. No points off if you didn't push through to the final solution.

   b. [3 pts] For your choice in part a, calculate the compliance $C_{xy}$, which relates the force in the y direction to the displacement in the x direction. $x = C_{xy} F_y$

$$F_{2x} = F_x,\quad M_2 = M_0,\quad F_{2y} = F_y$$

$$F_{1x} = F_x \qquad M_1 = M_0 + F_x L_2$$

$$F_{1y} = F_y$$



$$M_1 = 0.\ Angular\ deflection\ comes\ from\ Fy.\ \theta_1 = \frac{F_y {L_1}^2}{2EI} \quad \text{(1 pt)}$$

$$No\ deflection\ of\ L2\ (stays\ straight)$$

$$\Delta x = L_2 \tan(\theta_1) \approx L_2\theta_1 = L_2 \frac{F_y {L_1}^2}{2EI} \quad \text{(1 pt)}$$

$$C_{xy} = \frac{F_y}{\Delta x} = \frac{2EI}{{L_1}^2\, L_2} \quad \text{(1 pt)}$$

c. [2 pts] For your choice in part a, calculate the compliance $C_{yx}$ and explain what it means.

$$F_x \rightarrow \Delta y$$

$$M_{beam1}(L_1) = F_x L_2 \qquad \text{(1pt)}$$

$$\frac{M}{EI} = \frac{d^2 y_{beam1}}{dx^2} = \frac{F_x L_2}{EI}$$

$$\frac{dy_{beam1}}{dx} = \frac{F_x L_2}{EI} x + C_1$$

$$y_{beam1}(x) = \frac{F_x L_2}{2EI} x^2 + C_1 x + C_2$$

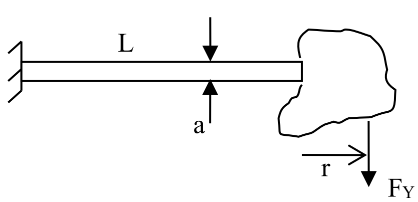
$$Boundary\ Conditions:$$

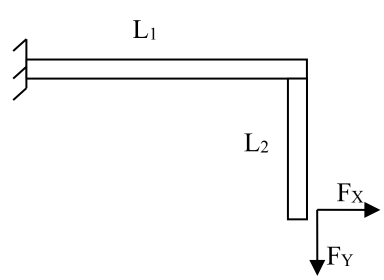
$$Fixed\ end - \text{y(0)=0 and y'(0)=0}$$

$$Thus\ y_{beam1}(x) = \frac{F_x L_2}{2EI} x^2 = \frac{6F_x L_2}{Eab^3} x^2$$

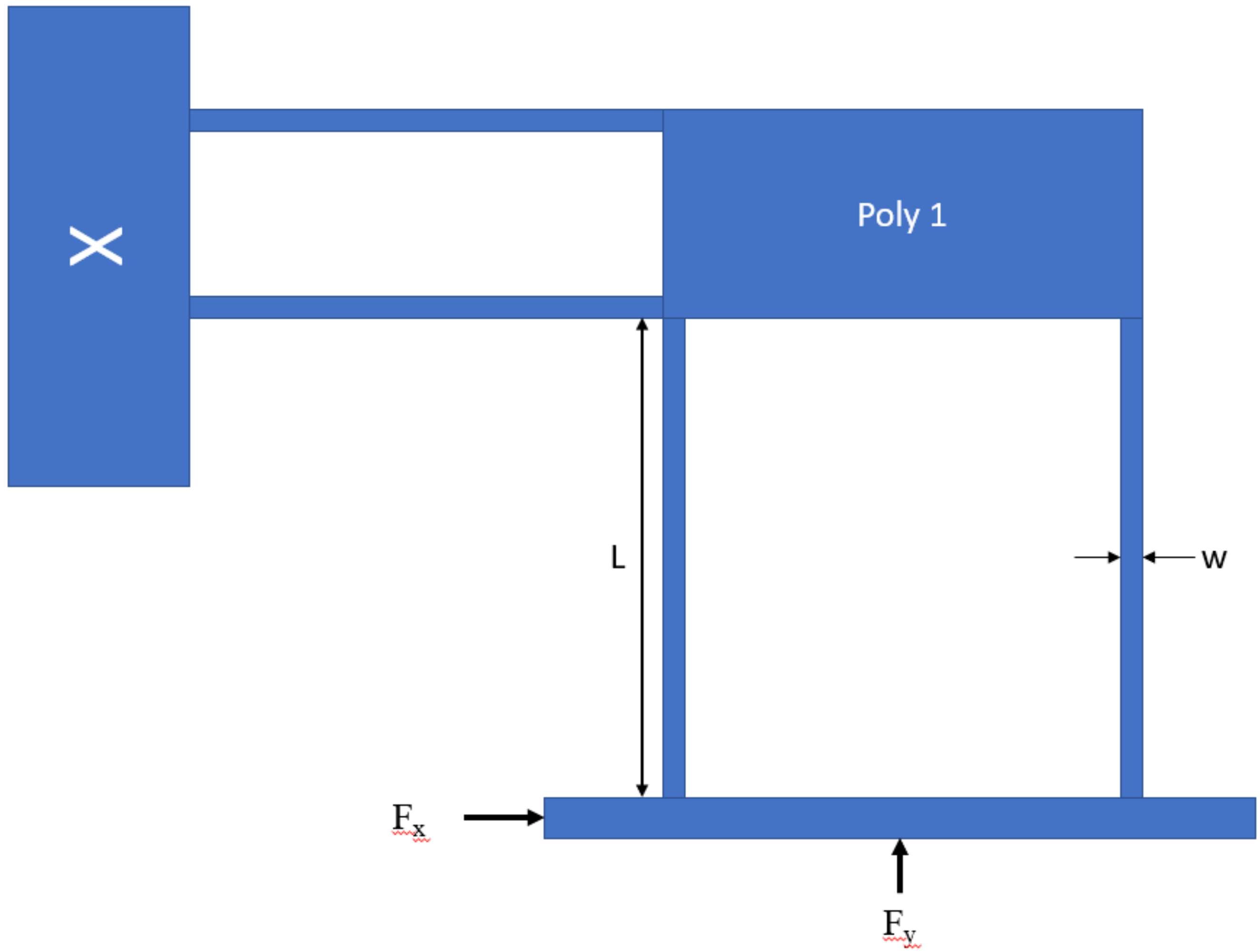
$$y_{beam1}(L_1) = \frac{6F_x L_2}{Eab^3}(L_1)^2 \qquad \text{(1pt)}$$

It's fine if you jumped straight to $y = M{L_1}^2/2EI$




4. [4 pts] Design a suspension in POLYMUMPS to have a stiffness of 1 N/m in x and y with no cross-coupling.




$The\ X\ denotes\ an\ anchors, while\ the\ rest\ of\ the\ structure\ is\ free. Design\ spring\ constants\ to\ be\ 1\frac{N}{m}$

Poly 1 thickness is fixed at 2um and the minimum width is also 2um. Approximating E=150GPa

$k_x = k_y$ if the 4 beams are identical

Because the axial stiffness of a beam is much larger than its stiffness due to a transverse force.

Thus, the vertical beams are essentially rigid due to $F_y$ so the stiffness comes from the horizontal beams in the y direction. The same is true for the beams in the x direction so

$k_x = k_y$

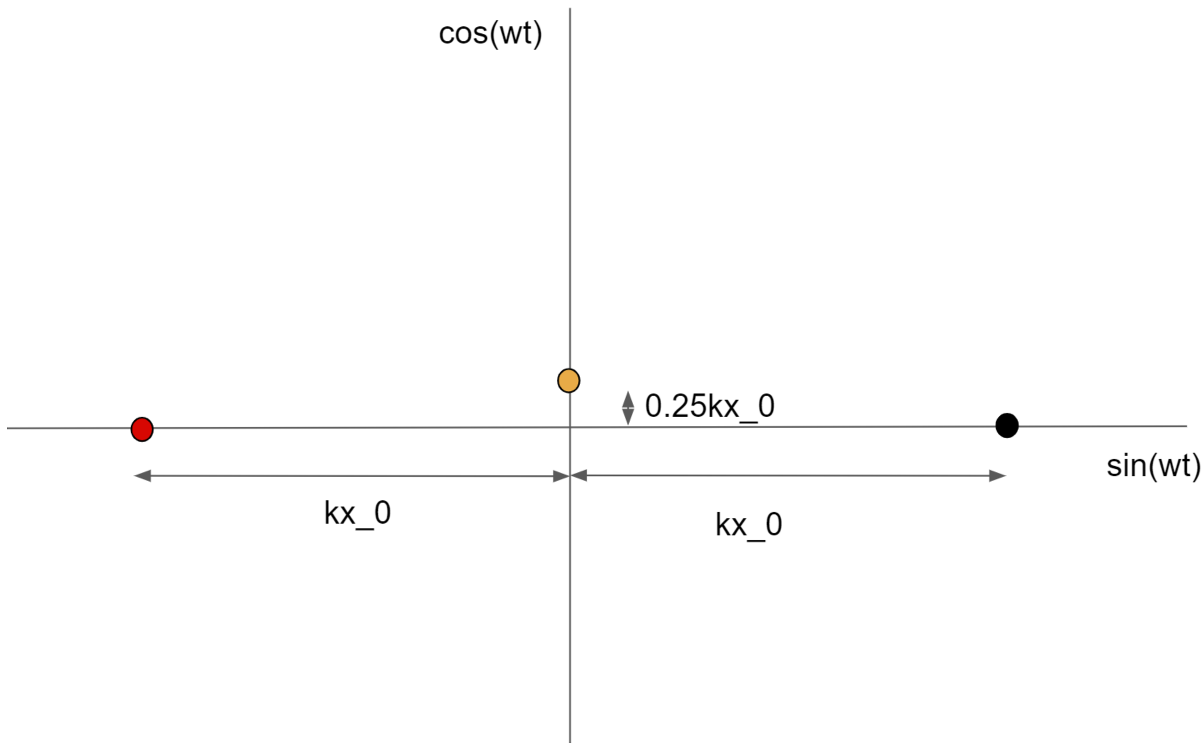
$Since\ there\ are\ 2\ beams\ in\ parallel\ for\ each\ direction, each\ beam\ needs\ to\ have\ a\ stiffness\ of\ 0.5\frac{N}{m}$. Stiffness of a clamped-clamped beam is 4 times the stiffness of a cantilever of the same length.

$$k_{x,one\ beam} = k_{y,one\ beam} = \frac{Etw^3}{L^3} = 0.5\frac{N}{m} = \frac{150G\ Pa(2um)(2um)^3}{L^3} \rightarrow L = 100um$$

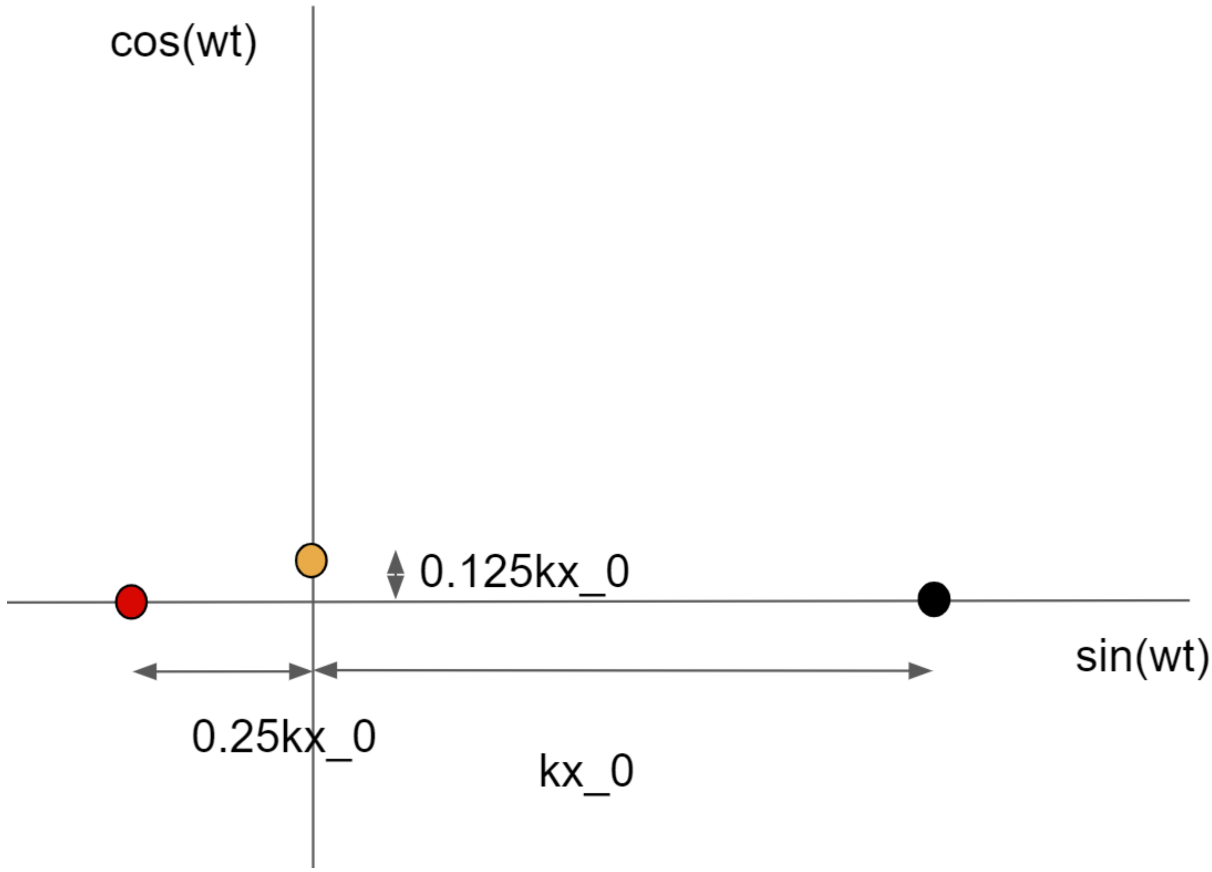
5. [247] [8 pts] In the structure on the right above, is it possible to attach a rigid body to the end of L2 and choose the point of action of the two forces such that $C_{xy}=C_{yx}=0$? If so, sketch your design.

6. [23 points total]

   a. [3 pts] Black is the spring, red is the mass and orange is damping. Must have magnitude labeled correctly for credit. Full credit if on the inertial term you wrote $-m\omega_n^2x_0$ or $-m\omega^2x_0$ instead of kx0 .

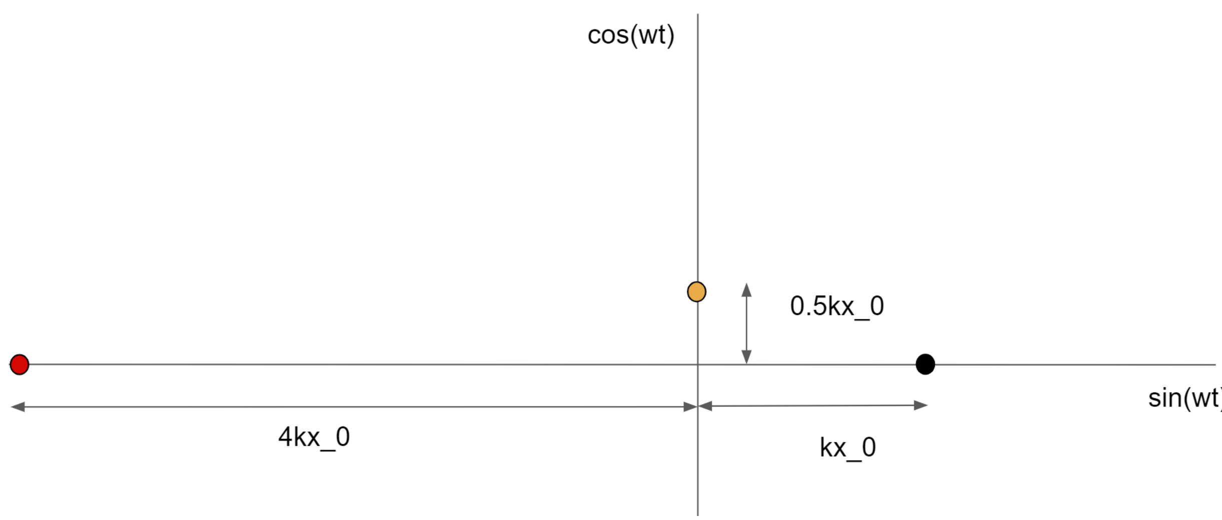



   b. [2 pts] The resulting force is $0.25kx_0\cos(w_nt)$. 1 pt for magnitude, 1 pt for phase (i.e. cos)
   c. [5 pts] 1 pt for getting kx0, 1 pts each for plotting on the correct axis for the damping and inertial forces, and an additional point each for getting the correct magnitude.



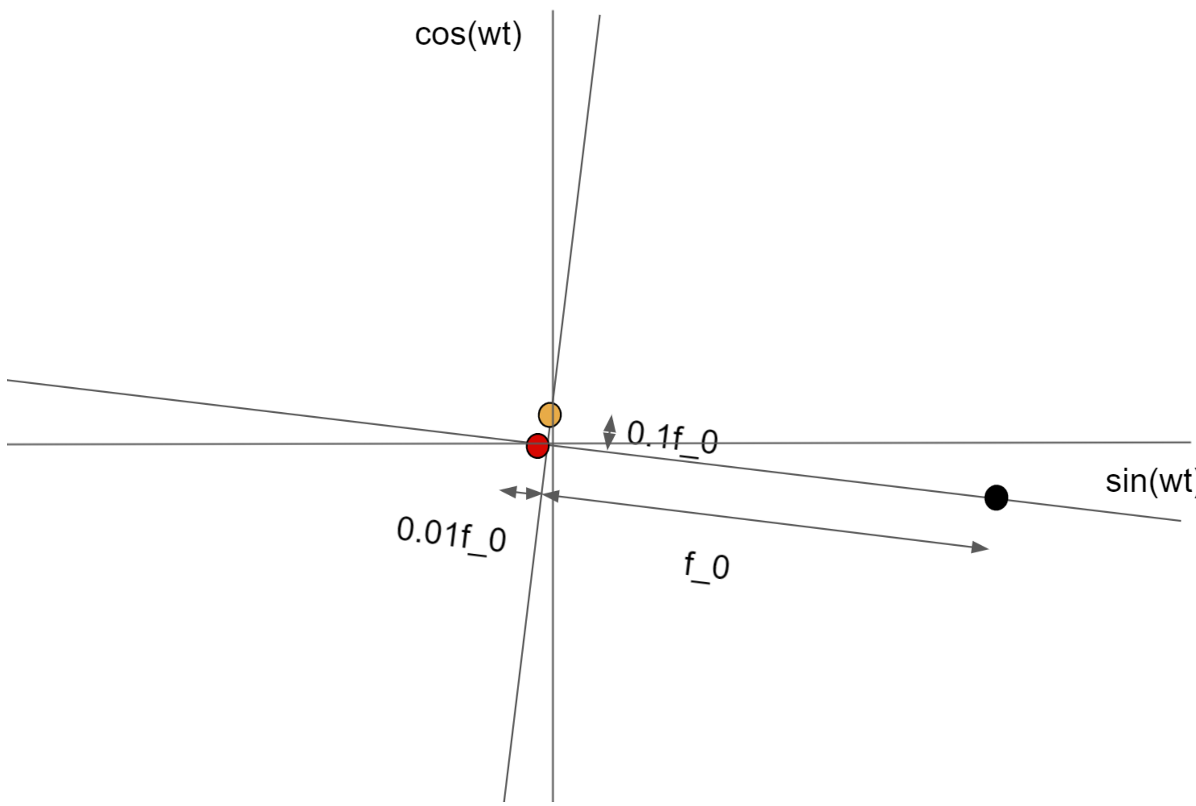

d. [4pts] 0.75 $kx_0 \sin(\omega_n t/2)$ + 0.125 $kx_0 \cos(\omega_n t/2)$. 1 pt each for magnitude and phase of each term. If you didn't write the sine and cosine, but wrote 0.76 kx0, that's 2 points. Full credit for magnitude and phase, 0.76 $\sin(\omega_n t/2+0.17)$. Fine to have phase in degrees (9.5 degrees)

e. [5 pts] same as part c.

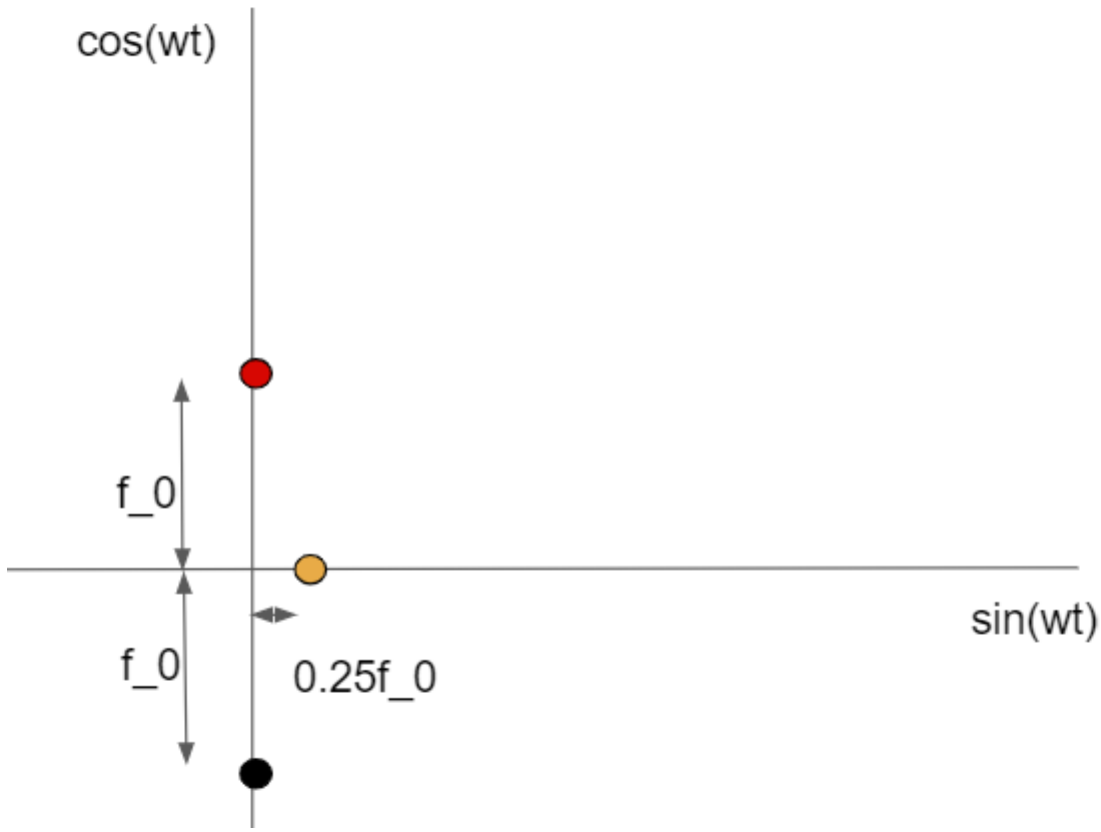


f. [4pts] -3 $kx_0 \sin(2\omega_n t)$ + 0. 5 $kx_0 \cos(2\omega_n t)$. 1 pt each for magnitude and phase of each term. If you didn't write the sine and cosine, but wrote 3.04 kx0, that's 2 points. Full credit for magnitude and phase, 3.04 $\sin(2\omega_n t+2.98)$. Fine to have phase in degrees (171 degrees)

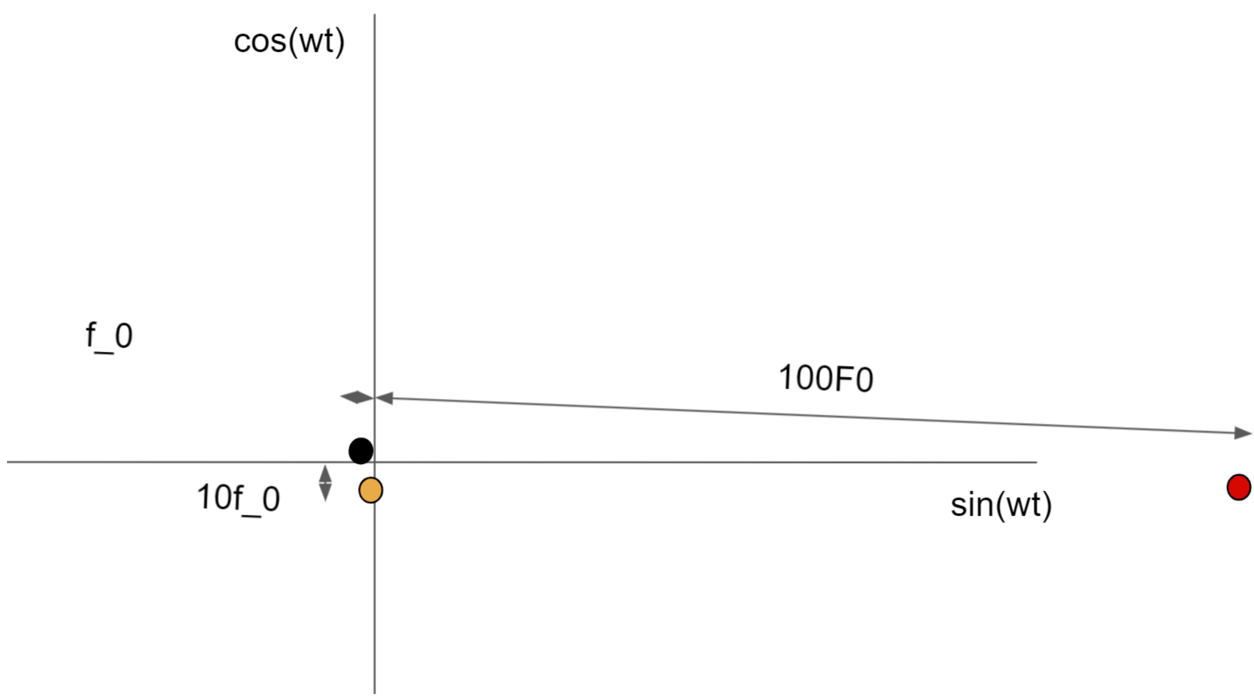
7. [15 points total]
   a. [4 pts] 1 each for magnitudes, 1 for approximate angle



b. [4 pts] Magnitudes below are wrong – and should all be 4 times bigger



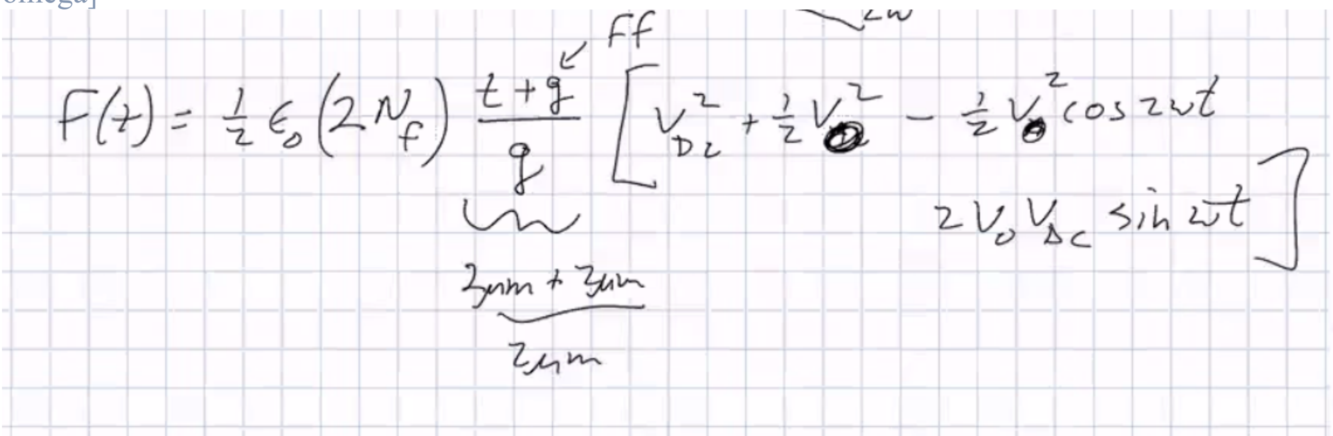

c. [4 pts] Magnitudes should be f0 for the inertial term, f0/40 for damping, f0/100 for spring. Angle should be just shy of 180.



d. [3 pts] At $\omega_n/10$ phase is -0.1 rad, or -6 degrees. At $\omega_n$ phase is -pi/2 rad, or -90 degrees. At $10\omega_n$ phase is –(pi-0.1) rad = -3.04 rad, or –(180-6) degrees = -174 degrees.

8. [9 pts] For a comb drive resonator with K=1 N/m, w=1krad/s, Q=10, and N=10 comb fingers, calculate the amplitude, phase, and frequency of all components of the deflection due to a 15 V 1 krad/s AC signal applied to one comb and a 15V DC signal applied to the body of the resonator

[1 pt each for amplitude, phase, frequency of each of three frequencies, DC, omega, 2 omega]




15V AC goes from -15V to 15V.

a. Forces at 0Hz, 1kHz and 2kHz

b.

At resonance → phase= $-90^o$ x(t) from Force and $X_{resonance} = QX_{DC}$

$$F_{comb} = \frac{N_{fingers}\epsilon_0 V^2(t+g)}{g} = \frac{10(8.85*10^{-12}\frac{F}{m})V^2(2um+2um)}{2um} = 1.8*10^{-10}\frac{F}{V^2}$$

$$X_{DC} = \frac{F_{comb}}{k}$$

$$X_{resonance} = QX_{DC} = Q\frac{F_{comb}}{k} = \frac{10}{1N/m}*1.8*10^{-10}\frac{F}{V^2} = 1.8*10^{-9}\frac{m}{V^2}$$

0Hz: $F = 6*10^{-7}m$

1kHz: $F = 2*10^{-7}\cos\left(2{,}000\frac{rad}{sec}*t-90^o\right)$

2kHz: $F = 8.1*10^{-7}\left(1{,}000\frac{rad}{sec}*t-90^o\right)$

c.

$$x_{res}(t) = 1.8*10^{-9}\left(337.5-112.5\cos\left(2{,}000\frac{rad}{sec}*t-90^o\right)+450\sin\left(1{,}000\frac{rad}{sec}*t-90^o\right)\right)m$$

x(t)=F(t) because k=1.

What the difference between sin(r) and -cos(r-$90^o$)?