1. b is the thickness of the poly1 layer, 2μm. a is the width of the beams, 2μm. L is the length of the beams, 300μm.
   a. **1 pt. for effort**
      1 pt. for an approximately right answer

$$k = \frac{Eba^3}{4L_c^3} = \frac{2Eba^3}{L^3}$$

$$k = \frac{2\left(150\times10^9\,\frac{\mathrm{N}}{\mathrm{m}^2}\right)(2\times10^{-6}\mathrm{m})^4}{(300\times10^{-6}\mathrm{m})^3}$$

$$\boldsymbol{k = 0.2\,\frac{\mathbf{N}}{\mathbf{m}}}$$

   b. **1 pt. for effort**
      1 pt. for an approximately right answer

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\boldsymbol{\omega_n = 60000\mathbf{s}^{-1} \cong 10\mathbf{kHz}}$$

c. **1 pt. for effort**
1 pt. for an approximately right answer for damping
1 pt. for an approximately right answer for Q

$$b = \frac{\mu A}{d}$$

$$= \frac{\left(2\times10^{-5}\,\frac{\mathrm{Ns}}{\mathrm{m}^2}\right)(100\times10^{-6}\mathrm{m})^2}{2\times10^{-6}\mathrm{m}}$$

$$\boldsymbol{b = 10^{-7}\,\frac{\mathbf{Ns}}{\mathbf{m}}}$$

$$Q = \frac{k}{b\omega_n}$$

$$Q = \frac{0.2\,\frac{\mathrm{N}}{\mathrm{m}}}{\left(10^{-7}\,\frac{\mathrm{Ns}}{\mathrm{m}}\right)(60000\mathrm{s}^{-1})}$$

$$\boldsymbol{Q = 30}$$

d. **1 pt. for effort**
1 pt. for an approximately right answer
For a constant voltage

$$i = \frac{d(CV)}{dt} = V_{DC}\frac{dC}{dt} = V_{DC}\frac{\partial C}{\partial x}\frac{\partial x}{\partial t}$$

$$i = V_{DC}\left(N_{gaps}\frac{\epsilon_0 h}{g}\right)\dot{x}$$

$$i = V_{DC}\left(N_{gaps}\frac{\epsilon_o h}{g}\right)\omega x_o \sin(\omega t)$$

$$i = \frac{(10\mathrm{V})(100)\left(10^{-11}\,\frac{\mathrm{F}}{\mathrm{m}}\right)(2\mu\mathrm{m})}{2\mu\mathrm{m}}\big(2\pi(1\mathrm{kHz})\big)(1\mu\mathrm{m})\sin(\omega t)$$

$$\boldsymbol{i(t) = 2\pi\times10^{-11}\mathbf{A}\,\sin(2\pi(1\mathbf{kHz})t)}$$

2. **1 pt. for each scaling relation (2 pts. total)**

   a. We know $\omega_n \propto \sqrt{k}$. k is proportional to the cube of the beam width, so $\omega_n \propto a^{3/2}$. If a is scaled by (1+x), then ω_n is scaled by $(1 + x)^{3/2}$. Doing a Taylor expansion for small x, we get that ω_n scales by $1 + \frac{3}{2}x$. This comes out to a 15% increase if x=0.1

   b. k is proportional to E, so $\omega_n \propto \sqrt{E}$. Doing the same analysis, $(1 + x)^{1/2} \rightarrow 1 + \frac{1}{2}x$. This is a 5% increase for x=0.1.

3. **1 pt. for stating the linear relationship between $\boldsymbol{a_{drawn}}$ and $\boldsymbol{\omega_n^{2/3}}$**
1 pt. for your explanation using the plot of $\boldsymbol{a_{drawn}}$ vs. $\boldsymbol{\omega_n^{2/3}}$ (You don't need to plot anything. Just state that you are going to use it)
1 pt. for relating Young's Modulus to the slope
1 pt. for relating $\boldsymbol{a_{offset}}$ to the x-intercept

We can relate the resonant frequency to beam width, a by

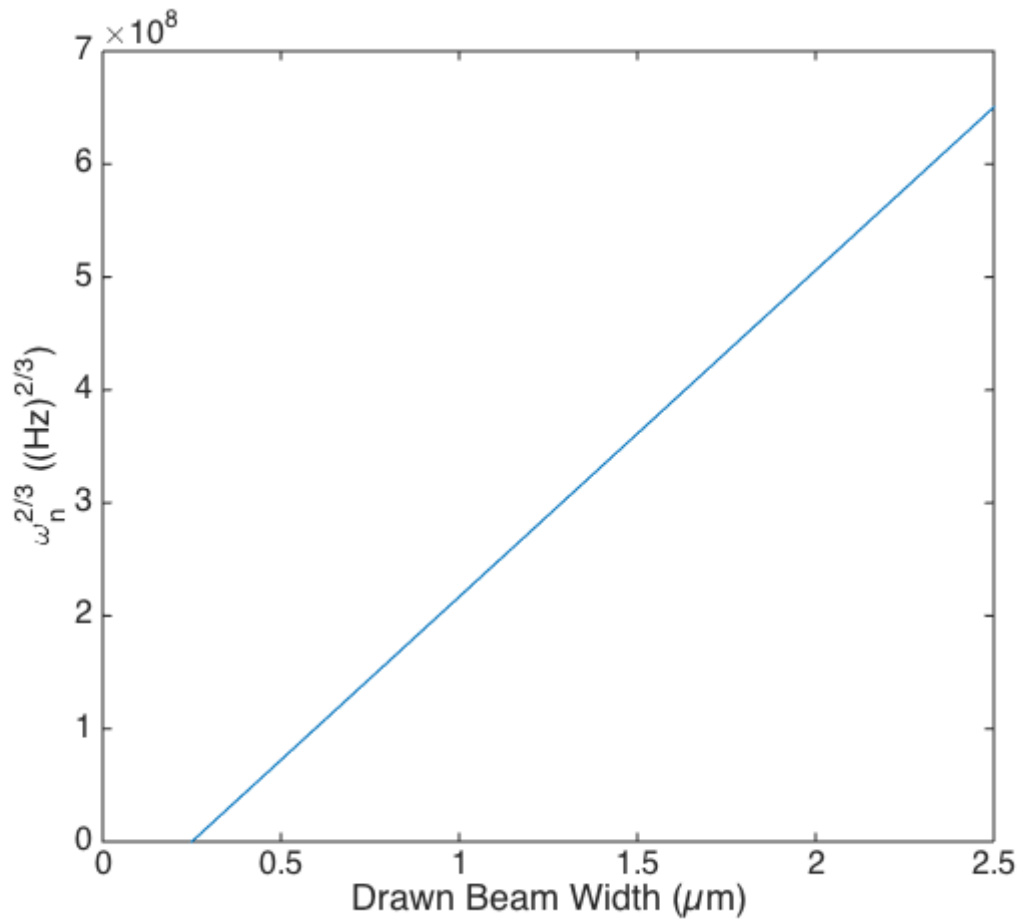
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2Eba^3}{mL^3}}$$

$$\omega_n = \gamma a^{\frac{3}{2}}$$

Where $\gamma = \sqrt{\frac{2Eb}{mL^3}}$ is a constant. Let us assume that we have measured the resonant frequency of each structure. If we plot the values of a vs. the measured frequencies ω_{meas} we would expect for the resonant frequency to go to zero as a goes to zero. This is usually not the case, as structures hardly ever come out in fabrication as drawn. There is always some offset between what is drawn on a mask and what your lithography actually does. What we actually get is

$$\omega_{meas} = \gamma\left(a_{drawn} + a_{offset}\right)^{\frac{3}{2}}$$

Now if we plot a_{drawn} vs. $\omega_{meas}^{\frac{2}{3}}$ we will get a linear plot where the extrapolated x-intercept will be the negative of the beam width offset. The plot below shows a linear extrapolation of what would happen if the beams were ~0.25μm smaller than drawn (The resonant frequency is lower than expected). The Young's modulus can be found from the slope of the line which is $\gamma^{\frac{2}{3}}$.



×10^8
7
6
5
4
3
2
1
0
ω_n^{2/3} ((Hz)^{2/3})
0
0.5
1
1.5
2
2.5
Drawn Beam Width (μm)

4. **1 pt. for each scaling relation (8 pts. total).**
Assuming we keep the substrate gap at 2μm

$$k \propto \frac{hw^3}{L^3} \rightarrow k \propto S$$

$$m \propto V \propto S^3$$

$$b \propto A \propto S^2$$

$$F \propto \frac{h}{g} \rightarrow \text{Does not change}$$

$$\omega_n = \sqrt{\frac{k}{m}} \propto \sqrt{\frac{S}{S^3}} \rightarrow \omega_o \propto \frac{1}{S}$$

$$Q = \frac{k}{b\omega_o} \propto \frac{S}{S^2\left(\frac{1}{S}\right)} \rightarrow Does\ not\ change$$

$$x_{DC} = \frac{F}{k} \rightarrow \frac{1}{S}$$

$$x_{resonance} = \frac{F}{b\omega_o} \propto \frac{1}{S^2\left(\frac{1}{S}\right)} \rightarrow \frac{1}{S}$$

5. **1 pt. for each scaling relation (8 pts. total). It is fine if you scaled the substrate gap.**
   Assuming gap to the substrate stays the same

$$k \propto h \rightarrow k \propto S$$

$$m \propto h \propto S$$

$$b \rightarrow Does\ not\ change$$

$$F \propto h \propto S$$

$$\omega_n = \sqrt{\frac{k}{m}} \propto \sqrt{\frac{S}{S}} \rightarrow \omega_n \rightarrow Does\ not\ change$$

$$Q \propto k \propto S$$

$$x_{DC} = \frac{F}{k} \propto \frac{S}{S} \rightarrow Does\ not\ change$$

$$x_{resonance} \propto F \propto S$$

6. **1 pt. for effort**
   1 pt. for the approximately right answer

$$F = N_{gaps}\left(\frac{1}{2}\epsilon_o V_{DC}^2 \frac{h}{g}\right) = kx_{DC}$$

$$x_{DC} = \frac{N_{gaps}\left(\frac{1}{2}\epsilon_o V_{DC}^2 \frac{h}{g}\right)}{k}$$

$$= \frac{(100)(0.01\text{nN})}{0.2\ \text{N/m}}$$

$$x_{DC} = 5\text{nm}$$

7. **1 pt. for effort**
   1 pt. for getting the approximately right answer
   $$F = N_{gaps}\left(\frac{1}{2}\epsilon_o \frac{h}{g}\right)\left[V_{DC}^2 + \frac{1}{2}V_{AC}^2 + 2V_{DC}V_{AC}\sin(\omega t) - \frac{1}{2}V_{AC}^2\cos(2\omega t)\right]$$

   The ω component will be

   $$F_\omega = N_{gaps}\left(\frac{1}{2}\epsilon_o \frac{h}{g}\right)(2V_{DC}V_{AC})$$

   $$= N_{gaps}\left(\frac{1}{2}\epsilon_o (V_{DC})^2 \frac{h}{g}\right)\frac{2V_{AC}}{V_{DC}}$$

   $$= (100)(1\text{nN})\left(2\left(\frac{1}{10}\right)\right)$$

   $$F_\omega = 20\text{nN}$$

   Assuming we are far below resonance, we can use DC deflection

   $$x_{1Hz} = \frac{F_\omega}{k} = 100\text{nm}$$

8. **1 pt. for effort**
   1 pt. for the approximately right deflection amplitude in each section (3 pts. total)
   1 pt. for getting the phase right in each section (3 pts. total)

   @ DC the phase is zero

   $$x_{DC} = \frac{F_{DC}}{k} = \frac{N_{gaps}\left(\frac{1}{2}\epsilon_o \frac{h}{g}\right)\left(V_{DC}^2 + \frac{1}{2}V_{AC}^2\right)}{k}$$

   $$x_{DC} \cong \frac{N_{gaps}\left(\frac{1}{2}\epsilon_o \frac{h}{g}V_{DC}^2\right)}{k}$$

   $$x_{DC} = \frac{(100)(1\text{nN})}{0.2\,\frac{\text{nN}}{\text{nm}}}$$

   $$x_{DC} = 500\text{nm}$$

@ ω_o the phase is -90^o

$$x_{\omega_n} = \frac{F_\omega}{b\omega_n} = \frac{N_{gaps}\left(\frac{1}{2}\epsilon_o\frac{h}{g}\right)2V_{DC}V_{AC}}{b\omega_n}$$

$$x_{\omega_n} = \frac{20\text{nN}}{\left(10^{-7}\frac{\text{Ns}}{\text{m}}\right)(60\times10^3\text{s}^{-1})}$$

$$x_{\omega_n} = 3\mu\text{m}$$

@ $2\omega_o$ the phase is -180^o

$$x_{2\omega_n} = \frac{F_{2\omega}}{m\omega} = \frac{N_{gaps}\left(\frac{1}{2}\epsilon_o\frac{h}{g}\right)\left(\frac{1}{2}V_{AC}^2\right)}{m(2\omega_n)^2}$$

$$= \frac{\frac{1}{2}(100)(0.01\text{nN})}{(4.6\times10^{-11}\text{kg})(120\times10^3\text{s}^{-1})^2}$$

$$x_{2\omega_o} = 0.8\text{nm}$$

The full displacement function will be (phases in radians)

$$x(t) = 0.5\mu\text{m} + 3\mu\text{m}\sin\left(\omega_\text{n}\text{t} - \frac{\pi}{2}\right) - 0.0008\mu\text{m}\cos(2\omega_n t - \pi)$$