1. 1 pt. for effort
   1 pt. for an approximately right answer
   \[ C(x) = \epsilon_0 t L \left( \frac{1}{g_1 - x} + \frac{1}{g_2 + x} \right) \]

b. 1 pt. for effort
   1 pt. for an approximately right answer
   \[ \frac{dC}{dx} = \epsilon_0 t L \left( \frac{1}{(g_1 - x)^2} - \frac{1}{(g_2 + x)^2} \right) \]

c. 1 pt. for effort
   1 pt. for an approximately right answer
   \[
   AD = \left. \frac{dC}{dx} \right|_{x=0} \\
   = \frac{\epsilon_0 t L \left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right)}{L(2a + g_1 + g_2)} \\
   = \frac{\epsilon_0 t \left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right)}{(2a + g_1 + g_2)}
   \]

d. 1 pt. for effort
   1 pt. for an approximately right answer
   Take the derivative of the aerial density with respect to \( g_2 \), and solve for \( g_2 \) when the derivative is zero.
   \[
   \frac{d(AD)}{dg_2} = \frac{\epsilon_0 t \left( \frac{2}{g_2^3} \right)}{2a + g_1 + g_2} - \frac{\epsilon_0 t \left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right)}{(2a + g_1 + g_2)^2} = 0 \\
   \]
   \[
   \frac{2}{g_2^3(2a + g_1 + g_2)} - \frac{\left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right)}{(2a + g_1 + g_2)^2} = 0 \\
   \]
   \[
   \frac{\left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right)}{(2a + g_1 + g_2)^2} = \frac{2}{g_2^3(2a + g_1 + g_2)}
   \]
\[ g_2^3 - g_2 g_1^2 = 2g_1^2(2a + g_1 + g_2) \]

\[ g_2^3 - (3g_1^2)g_2 - 2g_1^3 - 4ag_1^2 = 0 \]

**Extra note (not critical for grading):**

Usually when designing comb fingers for maximum aerial density, it is convenient to make \( g_1 \) the minimum feature size allowed by the design rules and then have \( g_2 = \alpha g_1 \). If we do this, we get a new cubic equation in terms of \( \alpha \)

\[ \alpha^3 - 3\alpha - 2 - \frac{4a}{g_1} = 0 \]

In most cases, we also make the finger width equal to the finger spacing, \( a = g_1 \). This gives

\[ \alpha^3 - 3\alpha - 6 = 0 \]

We can plot this equation for ratios greater than 0

The only real root is \( \alpha = 2.35 \), which fits with the result in part e, below.

If \( g_1 \) is much larger than \( a \), then the limiting value of \( \alpha \) is \( \sqrt{3} \). If \( g_1 \) is much smaller than \( a \), then the value of \( \alpha \) approaches the cube root of \( 4a/g_1 \). So making the
bigger gap 2.35 times bigger than the smaller gap is going to be pretty close to optimal over a wide range of line vs. space values for the fingers and primary gap.

e. 1 pt. for optimal $g_2$

1 pt. for maximum aerial density

Numerically solving the above equation, with $g_1 = a = 2\mu m$, we get $g_2 = 4.7\mu m$ as the only real solution. The maximum aerial density is $3.4F/m^3$. 
2. 1 pt. for finding the desired capacitance at zero displacement (total length of required fingers)

1 pt. for finding the dimensions of the mass and total mass based on the finger length and number of fingers

1 pt. for finding the necessary spring constant and spring dimensions from the mass and max displacement requirements

1 pt. for sketch with all dimensions labeled

1 pt. for \( \frac{dC}{dx} \) at \( x = 0 \)

At \( x = 0 \), we want a capacitance of 1pF. We must find the number of finger units, \( N \), required for the accelerometer. Using the dimensions from hw1 and the optimal gap spacing found above

\[
C(0) = 10^{-12} F = N\varepsilon_0 t_{fingers} \left( \frac{1}{g_1} + \frac{1}{g_2} \right)
\]

\[
N = \frac{10^{-12} F}{\varepsilon_0 t_{fingers} \left( \frac{1}{g_1} + \frac{1}{g_2} \right)}
\]

\[
N = \frac{10^{-12} F}{(8.85 \times 10^{-12} F/m)(20 \times 10^{-6} m)(20 \times 10^{-6} m) \left( \frac{1}{2 \times 10^{-6} m} + \frac{1}{4.7 \times 10^{-6} m} \right)}
\]

\[
N \approx 396
\]

We will assume the mass is still 20\( \mu \)m in the finger-less dimension, \( W \). With 396 finger units, the total length of the mass will be

\[
L = 396(2a + g_1 + g_2)
\]

\[
= 396(2(2\mu m) + 2\mu m + 4.7\mu m)
\]

\[
L = 4200\mu m
\]

That gives us a really long and thin shape, which is fine for your solutions but not what anyone would do in practice. If the array is about 20\( \mu \)m in one dimension, and 4,000 in the other, we can increase the length of the fingers by 10x to 200\( \mu \)m,
and reduce the number of fingers by 10x to 40. Now we have a shape that is closer to square – a little over 200µm by a little over 400µm.

Neglecting the fingers (probably not a good approximation!), the mass will be

\[ m_{plate} = \rho V_{mass} \]
\[ = 2300 \text{ kg/m}^3 (423 \times 10^{-6}\text{m})(20 \times 10^{-6}\text{m})(20 \times 10^{-6}\text{m}) \]
\[ m = 4 \times 10^{-10}\text{kg} \]

The mass of the fingers is

\[ m_{fingers} = (2300 \text{ kg/m}^3)(40)(2e^{-4}\text{m})(2e^{-6}\text{m})(2e^{-5}\text{m}) \]
\[ = (2.3)(4)(2)(2) \times 10^{3+1-4-6-5} \text{ kg} \]
\[ = 7.4 \times 10^{-10} \text{kg} \]

So the total weight is dominated by the fingers, and is about \(10^{-9}\) kg.

At \(2g\), we want an approximate deflection of the finger gap, 2µm. Equating the force equations for spring deflection and acceleration, we get

\[ kx = ma = m(2g) \]
\[ k = \frac{m(2g)}{x} \]
\[ k = \frac{(10^{-9}\text{kg})(2(10\text{m/s}^2))}{2 \times 10^{-6}\text{m}} \]
\[ k = 0.01 \text{ N/m} \]

We must find the beam length necessary for this given spring constant. For a spring that is 2µm wide and 20µm high, we get

\[ k = \frac{Ehw^3}{4L_s^3} \]
\[ L_s = \sqrt[3]{\frac{Ehw^3}{4k}} \]
\[ L_s = \sqrt[3]{\frac{150 \times 10^9\text{N/m}^2(20 \times 10^{-6}\text{m})(2 \times 10^{-6}\text{m})^3}{4(0.01\text{N/m})}} \]
\[ L_s = 1\text{mm} \]
Note that this diagram and the mass calculation do not account for etch holes to release the mass.

At $x = 0$,

$$\frac{dC}{dx} = Ne_0 tL \left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right)$$

$$= (396) \left( 8.85 \times 10^{-12} \frac{F}{m} \right) (20)(20)(10^{-12} m) \left( \frac{1}{(2 \times 10^{-6} m)^2} - \frac{1}{(4.7 \times 10^{-6} m)^2} \right)$$

$$= 3 \times 10^{-7} \frac{F}{m}$$
3. For each cross section:
   1 pt. for an attempt
   1 pt. for including the correct layers in the correct order. From bottom to top, substrate, nitride, poly0, poly1 (the problem did not explicitly state that the oxide on top of poly0 has been etched, so if you included the oxide, you are fine)
   1 pt. for getting the cross sections correct (don't worry about getting the conformal depositions technically correct)

Top cross section

Bottom cross section
5. **5 pts. total.** Again, no straightforward solution. Reward yourself for appropriate effort.