Non-Convex Optimization through Sequential Convex Programming

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Non-Convex Optimization

- Reminder: Convex optimization:

  \[
  \min_x f_0(x) \\
  \text{s.t. } f_i(x) \leq 0 \ \forall i \\
  A(j,:)x - b_j = 0 \ \forall j
  \]

- Non-convex optimization:

  \[
  \min_x g_0(x) \\
  \text{s.t. } g_i(x) \leq 0 \ \forall i \\
  h_j(x) = 0 \ \forall j
  \]

  with:
  
  - \(g_i\) non-convex
  - \(h_j\) nonlinear
Sequential Convex Programming

To solve:

\[
\begin{align*}
\min_x g_0(x) \\
\text{s.t. } g_i(x) &\leq 0 \quad \forall i \\
h_j(x) &= 0 \quad \forall j
\end{align*}
\]  

(1)

and increase \( \mu \) in an outer loop until the two sums equal zero.

Solve:

\[
\begin{align*}
\min_x g_0(x) + \mu \sum_i |g_i(x)|^+ + \mu \sum_j |h_j(x)| &= \min_x f_\mu(x) \\
\text{merit function}
\end{align*}
\]  

(2)

To solve (2), repeatedly solve the convex program:

\[
\begin{align*}
\min_x g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+ + \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \\
\text{s.t. } ||x - \bar{x}||_2 &\leq \varepsilon \quad \text{(trust region constraint)} \\
\bar{x}: \text{ current point}
\end{align*}
\]
Inputs: $\bar{x}, \mu = 1, \varepsilon_0, \alpha \in (0.5, 1), \beta \in (0, 1), t \in (1, \infty)$

While $(\sum_i |g_i(\bar{x})|^+ + \sum_j |h_j(\bar{x})| \geq \delta$ AND $\mu < \mu_{\text{MAX}})$

$\mu \leftarrow t \mu, \quad \varepsilon \leftarrow \varepsilon_0$  // increase penalty coefficient for constraints; re-init trust region size

While (1)  // [2] loop that optimizes

Compute terms of first-order approximations: $g_0(\bar{x}), \nabla_x g_0(\bar{x}), g_i(\bar{x}), \nabla_x g_i(\bar{x}), h_j(\bar{x}), \nabla_x h_j(\bar{x}), \forall i, j$

While (1)  // [3] loop that does trust-region size search

Call convex program solver to solve:

$$(\bar{f}_\mu(\bar{x}_{\text{next?}}), \bar{x}_{\text{next?}}) = \min_x \quad g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+$$

$$+ \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \quad \text{s.t.} \quad \|x - \bar{x}\|_2 \leq \varepsilon$$

If $\frac{f_\mu(\bar{x}) - f_\mu(\bar{x}_{\text{next?}})}{f_\mu(\bar{x}) - f_\mu(\bar{x}_{\text{next?}})} \geq \alpha$

Then shrink trust region:

Else Update $\bar{x} \leftarrow \bar{x}_{\text{next?}}$, Grow trust region: $\varepsilon \leftarrow \varepsilon / \beta$, and Break out of while [3]

If below some threshold, Break out of while [3] and while [2]
Non-Convex Optimization

- Non-convex optimization with convex parts separated:

\[
\min_x f_0(x) + g_0(x)
\]

s.t. \[
\begin{align*}
    f_i(x) & \leq 0 \quad \forall i \\
    Ax - b &= 0 \quad \forall j \\
    g_k(x) & \leq 0 \quad \forall k \\
    h_l(x) &= 0 \quad \forall l
\end{align*}
\]

with:

- \( f_i \) convex
- \( g_k \) non-convex
- \( h_l \) nonlinear

- Retain convex parts and in inner loop solve:

\[
\min_x f_0(x) + g_0(x) + \mu \sum_k |g_k(x)|^+ + \mu \sum_l |h_l(x)|
\]

s.t. \[
\begin{align*}
    f_i(x) & \leq 0 \quad \forall i \\
    Ax - b &= 0 \quad \forall j
\end{align*}
\]